Orthogonal polynomials and discrete Painlevé equations

Walter Van Assche

Katholieke Universiteit Leuven, Belgium

orthogonal polynomials on the real line

Orthonormal polynomials on the real line are defined by

$$\int_{\mathbb{R}} p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n},$$

where μ is a positive measure on the real line.

They always satisfy a three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n \in \mathbb{N}$$

with $p_0 = 1$, $p_{-1} = 0$ and $a_n > 0$, $b_n \in \mathbb{R}$

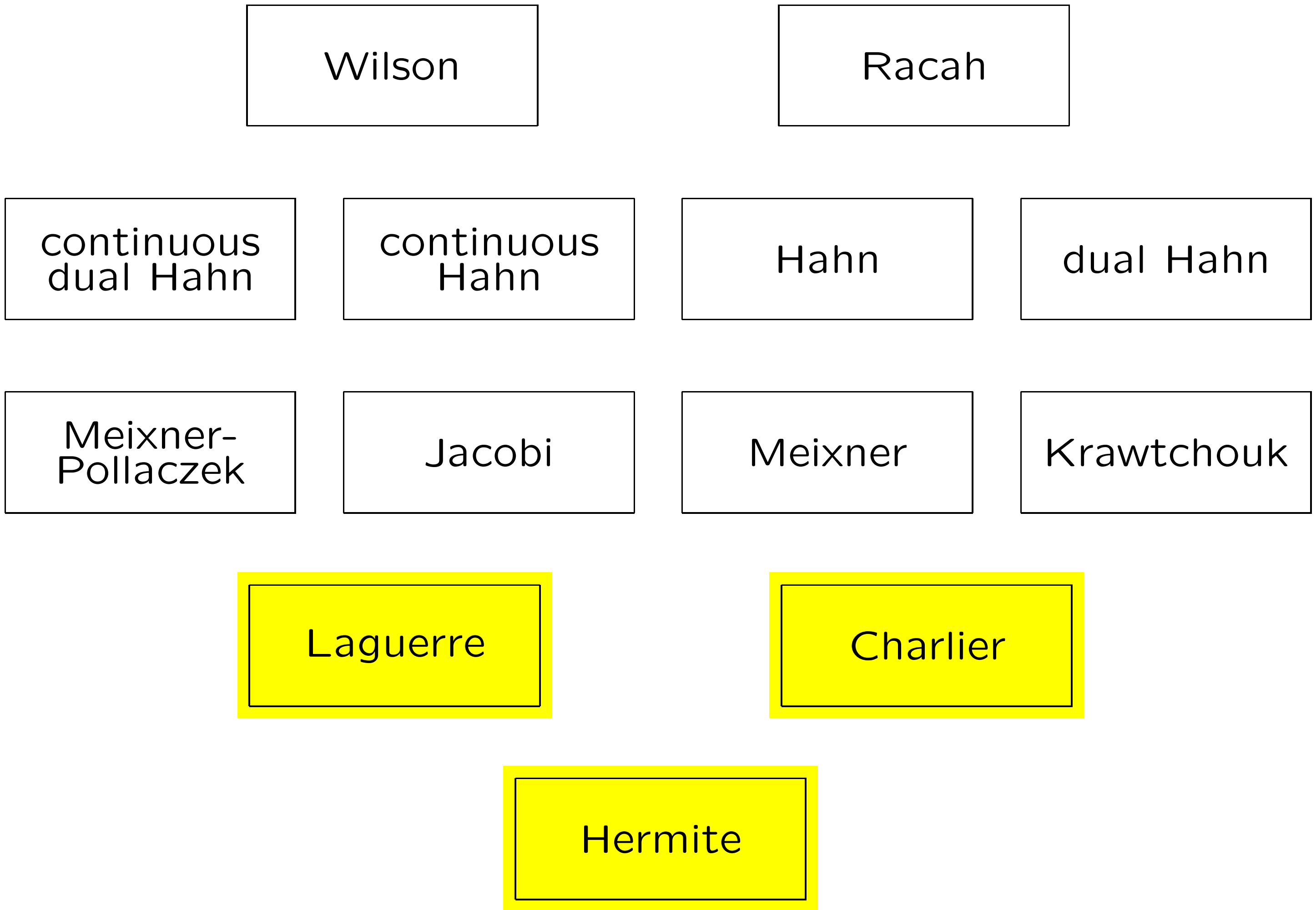
$$b_n = \int x p_n^2(x)\,d\mu(x), \qquad a_n = \int x p_n(x) p_{n-1}(x)\,d\mu(x)$$

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \qquad n \in \mathbb{N}$$

gives the Jacobi operator

$$J = \begin{pmatrix} b_0 & a_1 & & & & & 0 \\ a_1 & b_1 & a_2 & & & & \\ & a_2 & b_2 & a_3 & & & \\ & & a_3 & b_3 & a_4 & & \\ & & & a_4 & \ddots & \ddots & \\ 0 & & & & \ddots & & \end{pmatrix}$$

Askey tableau



N. Witte starts from the top (Askey-Wilson)

speaker et al. start from the bottom

Shohat, Freud non-linear recurrence

Its et al. recognize discrete Painlevé

Periwal, Shevitz unit circle

Magnus Freud vs Painlevé

Nijhoff, Spicer more semi-classical weights

Y. Chen, Ismail, et al. other nice examples

Classical weights satisfy the **Pearson equation**

$$D(\sigma w) = \tau w, \quad \deg \sigma \leq 2, \deg \tau = 1$$

$D = d/dx$ (derivative): Hermite, Laguerre, Jacobi

$D = \Delta$ (forward difference): Charlier, Krawtchouk + Meixner, Hahn

$\delta f(x) = f(x + i/2) - f(x - i/2)$: Meixner-Pollaczek, continuous Hahn

$\Delta_\lambda f(x) = \frac{\Delta f(\lambda(x))}{\Delta(\lambda(x))}$ (divided difference, λ quadratic): dual Hahn, Racah

$\delta_\lambda f(x) = \frac{\delta f(\lambda(x))}{\delta(\lambda(x))}$: continuous dual Hahn, Wilson

Consequence of the Pearson equation:

$$\sigma(x) D p_n(x) = \sum_{k=n}^{n+r} A_{n,k} p_{k-1}(x), \qquad r = \deg \sigma$$

Compatibility with the recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

gives explicit expressions for the recurrence coefficients when $r \leq 2$ (classical weights).

For $r > 2$ (semi-classical weights) we get non-linear recurrence relations

Freud weights

Exponential weights on $\mathbb{R}$:

$$w(x) = |x|^\rho \exp(-|x|^m), \qquad \rho > -1,\ m > 0$$

$m = 2$, $\rho = 0$ are the Hermite polynomials

$$x p_n(x) = \sqrt{\frac{n+1}{2}}\, p_{n+1}(x) + \sqrt{\frac{n}{2}}\, p_{n-1}(x)$$

What are the recurrence coefficients for Freud weights?

Symmetry: $w(-x) = w(x)$ implies that $b_n = 0$, $n \in \mathbb{N}$

Freud equations

Consider $m = 4$: $w(x) = |x|^\rho \exp(-x^4)$.

Important observation for $\rho = 0$: Pearson equation

$$\Big(\exp(-x^4)\Big)' = -4x^3 \exp(-x^4).$$

It implies the structure relation

$$p_n'(x) = A_n p_{n-1}(x) + B_n p_{n-3}(x)$$

compatibility between the recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_np_{n-1}(x) \qquad \boxed{xP = JP}$$

and the structure relation

$$p_n'(x) = A_np_{n-1}(x) + B_np_{n-3}(x) \qquad \boxed{P' = LP}$$

$$J = \begin{pmatrix} b_0 & a_1 & & & & & 0 \\ a_1 & b_1 & a_2 & & & & \\ & a_2 & b_2 & a_3 & & & \\ & & a_3 & b_3 & a_4 & & \\ & & & a_4 & \ddots & \ddots & \\ 0 & & & & \ddots & & \end{pmatrix} \qquad L = \begin{pmatrix} 0 & & & & & & 0 \\ A_1 & 0 & & & & & \\ 0 & A_2 & 0 & & & & \\ B_3 & 0 & A_3 & 0 & & & \\ 0 & B_4 & 0 & A_4 & \ddots & & \\ 0 & & \ddots & \ddots & \ddots & & \end{pmatrix}$$

Compatibility condition:

$$\boxed{JL - LJ = I}$$

$$JL - LJ = I$$

(n,n)-entry gives

$$a_{n+1}A_{n+1} - a_nA_n = 1 \quad \Rightarrow \quad a_nA_n = n.$$

$(n,n-2)$-entry gives

$$a_nA_{n-1} - a_{n-1}A_n = a_{n-2}B_n - a_{n+1}B_{n+1}$$

$(n,n-4)$-entry gives

$$a_nB_{n-1} - a_{n-3}B_n = 0 \quad \Rightarrow \quad B_n = 4a_na_{n-1}a_{n-2}.$$

Combined this gives

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2)$$

Freud equation for $w(x) = |x|^\rho \exp(-x^4)$:

$$n + \rho\Delta_n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2)$$

with $a_0 = 0$ and

$$\Delta_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

discrete Painlevé equations

d-P_{I} $$x_{n+1}+x_n+x_{n-1}=\frac{z_n+\gamma(-1)^n}{x_n}+\delta$$

d-P_{II} $$x_{n+1}+x_{n-1}=\frac{x_n z_n+\gamma}{1-x_n^2}$$

d-P_{IV} $$(x_{n+1}+x_n)(x_n+x_{n-1})=\frac{(x_n^2-\kappa^2)(x_n^2-\mu^2)}{(x_n+z_n)^2-\gamma^2}$$

d-P_{V} $$\frac{(x_{n+1}+x_n-z_{n+1}-z_n)(x_n+x_{n-1}-z_n-z_{n-1})}{(x_{n+1}+x_n)(x_n+x_{n-1})}$$

$$=\frac{[(x_n-z_n)^2-\alpha^2][(x_n-z_n)^2-\beta^2]}{(x_n-\gamma^2)(x_n-\delta^2)}$$

with $z_n=\alpha n+\beta$ and $\alpha,\beta,\gamma,\delta,\kappa,\mu$ constants.

choose $\alpha = 1$, $\beta = \rho/2$, $\gamma = -\rho/2$, $\delta = 0$ then $2a_n^2 = x_n$ satisfies

$$x_n(x_{n+1} + x_n + x_{n-1}) = n + \rho\Delta_n$$

discrete Painlevé I. We are interested in a positive solution of this non-linear recurrence

Freud's analysis

$$x_n^2 \le x_n(x_{n+1} + x_n + x_{n-1}) = n + \rho\Delta_n$$

hence

$$\frac{x_n}{\sqrt{n}} \quad \text{is bounded}$$

$$x_n(x_{n+1} + x_n + x_{n-1}) = n + \rho\Delta_n$$

$$A = \liminf_{n\to\infty} \frac{x_n}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{x_n}{\sqrt{n}} = B$$

Choose a subsequence n' such that $x_{n'}/\sqrt{n'} \to A$, then for $n' \to \infty$

$$1 \leq A(2B + A)$$

Choose a subsequence n'' such that $x_{n''}/\sqrt{n''} \to B$, then for $n'' \to \infty$

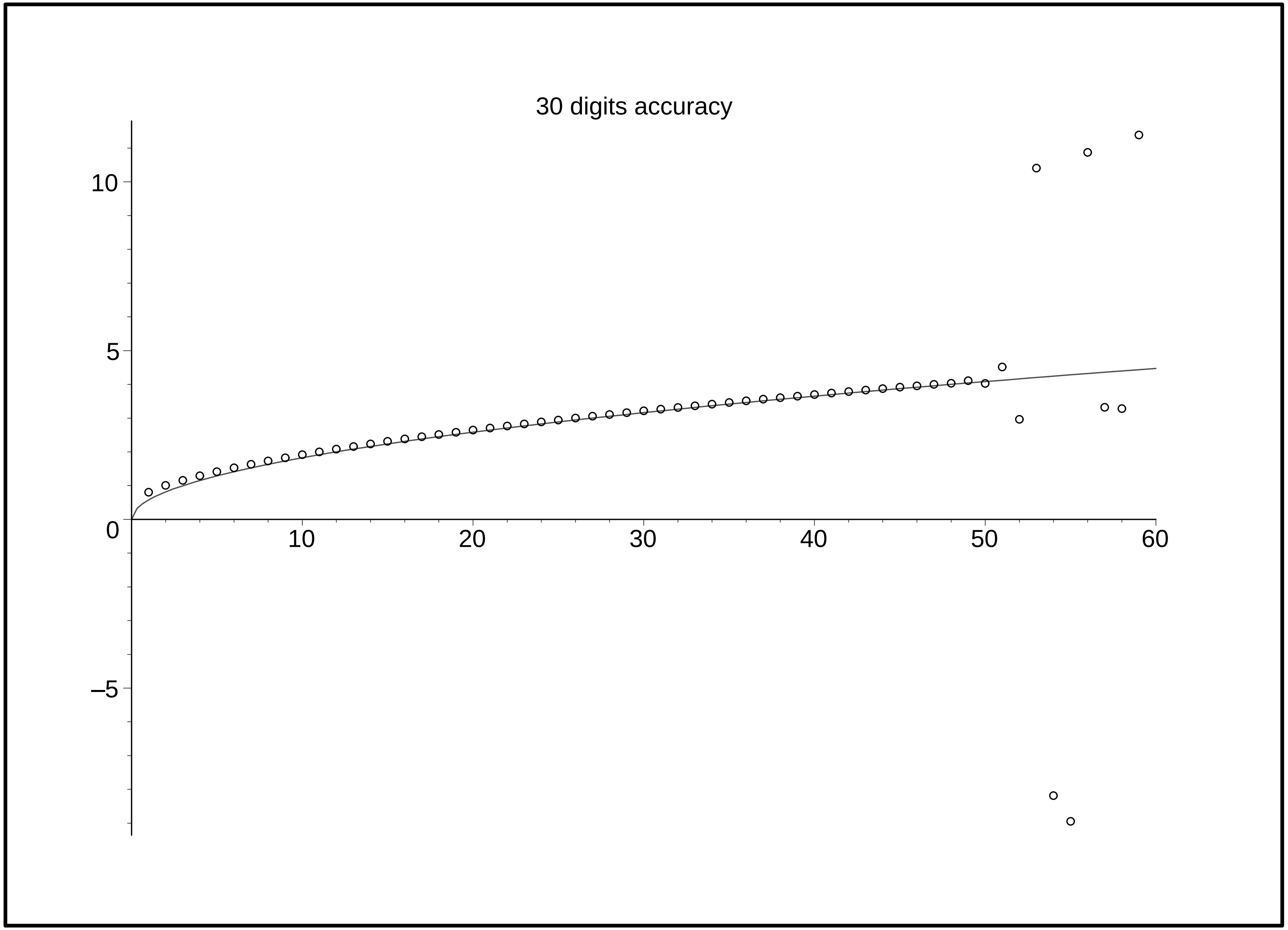
$$B(2A + B) \leq 1$$

Hence $2AB + B^2 \leq 2AB + A^2$ so that $B^2 \leq A^2$. But then $A = B$ and

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = A = B$$

exists. Take the limit in the recurrence relation, then $3A^2 = 1$, so that

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = \frac{1}{\sqrt{3}}, \qquad \lim_{n\to\infty} \frac{a_n}{n^{1/4}} = \frac{1}{12^{1/4}}.$$


30 digits accuracy
10
5
0
–5
10
20
30
40
50
60

$$x_n(x_{n+1} + x_n + x_{n-1}) = n + \rho\Delta_n$$

The recurrence relation is very unstable for computing the recurrence coefficients. Initial condition:

$$x_0 = 0, \quad x_1 = 2a_1^2 = \frac{2\int_{-\infty}^{\infty} x^2|x|^\rho e^{-x^4}\,dx}{\int_{-\infty}^{\infty} |x|^\rho e^{-x^4}\,dx} = \frac{2\Gamma(\frac{3+\rho}{4})}{\Gamma(\frac{1+\rho}{4})}$$

Lew and Quarles* showed that there is a unique positive solution of the recurrence relation with $x_0 = 0$.

Hence a small error in x_1 eventually destroys the positivity.

Nonnegative solutions of a nonlinear recurrence*, J. Approx. Theory **38 (1983), 357–379

Theorem 1 (Lew and Quarles) *The recurrence relation*

$$x_n(x_{n-1} + x_n + x_{n+1}) = \gamma_n^2,$$

with $x_0 = 0$ *has a unique positive solution if* $\lim_{n\to\infty} \gamma_n/n = 0$*.*

Proof: Operator $T : (x_0, x_1, x_2, \ldots) \mapsto (y_0, y_1, y_2, \ldots)$

$$\begin{aligned} y_0 &= 0 \\ y_n &= \frac{1}{2}\left(-(x_{n-1} + x_{n+1}) + \sqrt{4\gamma_n^2 + (x_{n-1} + x_{n+1})^2}\right), \qquad n \geq 1 \end{aligned}$$

Positive sequence with $x_0 = 0 \mapsto$ positive sequence with $y_0 = 0$.

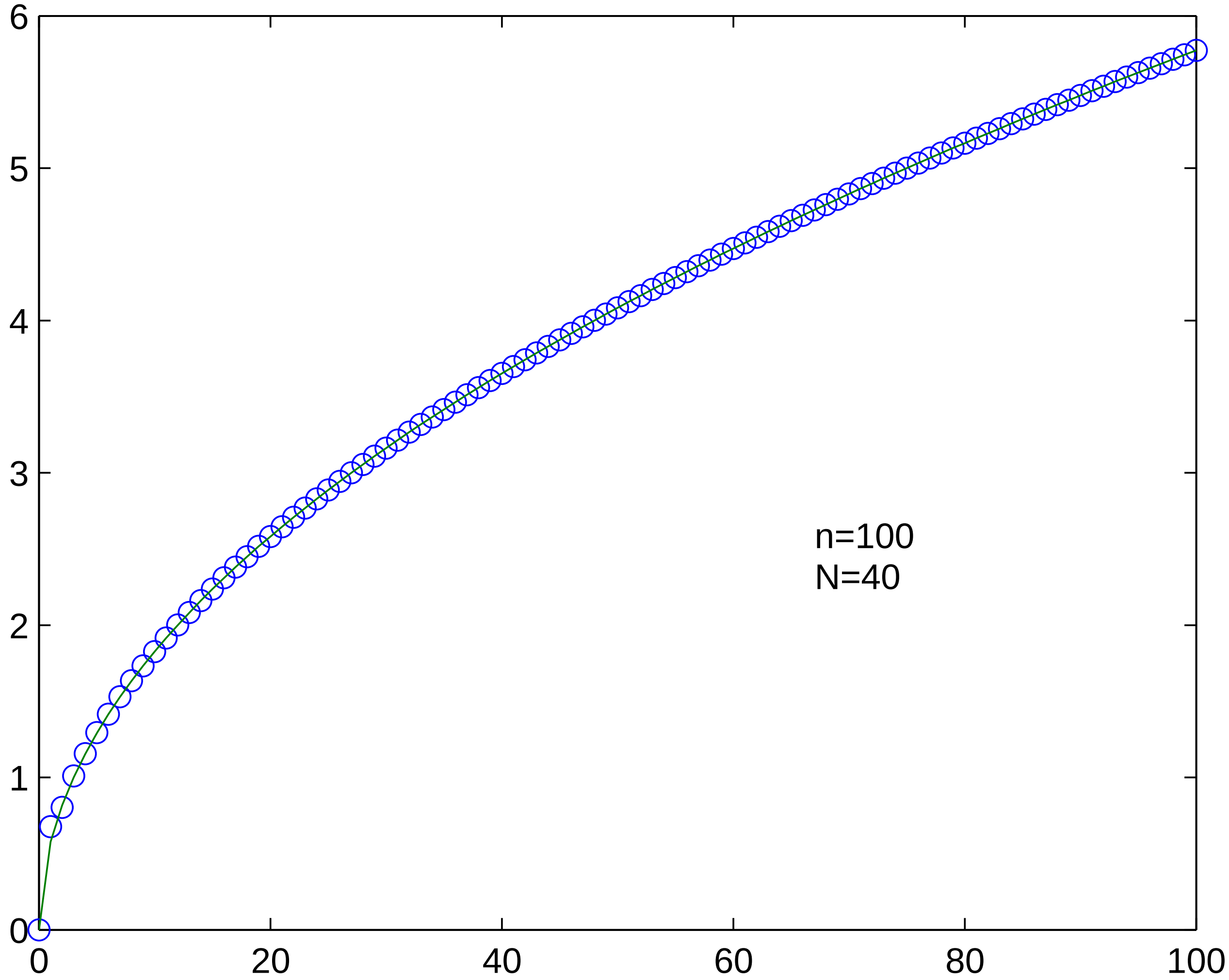
T^k is a contraction for some $k \Rightarrow T$ has a unique fixed point $Tx = x$.

This fixed point satisfies the non-linear recurrence relation.

This gives a stable way to compute the recurrence relation: iteration

computing $(x_0, x_1, \ldots, x_n)$
N is the number of iterations

```
x=[0,ones(1,n+N)];
for k=1:N
  y(1)=0;
  for j=2:n+N-k+1
    y(j)=(-(x(j+1)+x(j-1))+sqrt(4*(j-1)+(x(j+1)+x(j-1))^2))/2;
  end
  x=y(1:n+N-k+1);
end
```



6
5
4
3
2
1
0
0
20
40
60
80
100
n=100
N=40

Painlevé property

discrete Painlevé I

$$x_n(x_{n+1} + x_n + x_{n-1}) = n$$

discrete Painlevé property: **singularity confinement**

If x_n is such that it results in a singularity for x_{n+1}, then there exists a $p \in \mathbb{N}$ such that this singularity is confined to $x_{n+1}, x_{n+2}, \dots, x_{n+p}$. Furthermore x_{n+p+1} depends only on $x_{n-1}, x_{n-2}, \dots$.

discrete Painlevé I

$$x_n(x_{n+1} + x_n + x_{n-1}) = n$$

$$x_{n+1} = \frac{n}{x_n} - x_n - x_{n-1}$$

Singularities occur when $x_n = 0$:

$$\begin{aligned} x_n &= 0 \\ x_{n+1} &= \pm\infty \\ x_{n+2} &= \mp\infty \\ x_{n+3} &= \pm\infty \mp \infty = ?? \end{aligned}$$

More careful analysis: let $x_n = \epsilon$ and expand x_{n+k} in a Laurent series in ϵ

$$
\begin{aligned}
x_n &= \epsilon \\
x_{n+1} &= \frac{n}{\epsilon} - x_{n-1} - \epsilon \\
x_{n+2} &= -\frac{n}{\epsilon} + x_{n-1} + \frac{n+1}{n}\,\epsilon + \mathcal{O}(\epsilon^2) \\
x_{n+3} &= -\frac{n+3}{n}\,\epsilon + \mathcal{O}(\epsilon^2) \\
x_{n+4} &= \frac{n}{n+3}\,x_{n-1} + \mathcal{O}(\epsilon)
\end{aligned}
$$

Singularity confined to $x_{n+1}, x_{n+2}, x_{n+3}$.

discrete orthogonal polynomials

Orthogonality on $\mathbb{N} = 0, 1, 2, 3, \ldots$ with weights $w_k > 0$ $(k \in \mathbb{N})$

$$\sum_{k=0}^{\infty} p_n(k) p_m(k) w_k = \delta_{n,m}, \qquad n, m \geq 0$$

$w_k = \dfrac{a^k}{k!}$	Poisson distribution	Charlier polynomials
$w_k = \binom{N}{k} p^k (1-p)^{N-k}$	binomial distribution	Krawtchouk polynomials
$w_k = \dfrac{(\beta)_k c^k}{k!}$	negative binomial	Meixner polynomials
$w_k = \binom{\alpha + N}{k} \binom{\beta + N - k}{N - k}$	hypergeometric	Hahn polynomials

Charlier polynomials

$$w_k = \frac{a^k}{k!}, \qquad a > 0 \qquad w_{k-1} = \frac{k}{a}\, w_k$$

difference operators

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1).$$

$$\Delta p_n(k) = \frac{a_n}{a} p_{n-1}(k)$$

$$x p_n(x) = \sqrt{(n+1)a}\; p_{n+1}(x) + (n+a)\; p_n(x) + \sqrt{na}\; p_{n-1}(x)$$

generalized Charlier polynomials*

$$w_k = \frac{a^k}{(k!)^N}, \qquad N \in \mathbb{N} \qquad w_{k-1} = \frac{k^N}{a} w_k$$

For $N = 2$: structure relation

$$p_n(k+1) = p_n(k) + A_n p_{n-1}(k) + B_n p_{n-2}(k)$$

*M.N. Hounkonnou, C. Hounga, A. Ronveaux: *Discrete semi-classical orthogonal polynomials: generalized Charlier*, J. Comput. Appl. Math. **114** (2000), 361–366.

$$\Delta p_n(k) = \frac{n}{a_n}\, p_{n-1}(k) + \frac{a_n a_{n-1}}{a}\, p_{n-2}(k)$$

compatibility with

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

gives

$$-na(b_n - b_{n-1} - 1) = a_n^2 \left(a_{n+1}^2 - a_{n-1}^2\right)$$

$$a_n^2 (b_n + b_{n-1} - n + 1) = na$$

Put $a_n^2 = a(1-c_n^2)$, and $b_n = \hat{b}_n + n$ then

$$\hat{b}_n = \sqrt{a}\ c_n c_{n+1}$$

and

$$(1-c_n^2)\sqrt{a}(c_{n+1}+c_{n-1}) = nc_n$$

Theorem 2 *The asymptotic behavior is given by*

$$\lim_{n\to\infty} a_n^2 = a, \qquad \lim_{n\to\infty} b_n - n = 0$$

W. Van Assche, M. Foupouagnigni: *Analysis of non-linear recurrence relations for the recurrence coefficients of generalized Charlier polynomials*, J. nonlinear Math. Phys. **10**, supplement 2 (2003), 231–237

Put $a_n^2 = a(1 - c_n^2)$, and $b_n = \hat{b}_n + n$ then

$$\hat{b}_n = \sqrt{a}\ c_n c_{n+1}$$

and

$$(1 - c_n^2)\sqrt{a}(c_{n+1} + c_{n-1}) = nc_n$$

Theorem 2 *The asymptotic behavior is given by*

$$\lim_{n\to\infty} a_n^2 = a, \qquad \lim_{n\to\infty} b_n - n = 0$$

Proof: $a_n^2 > 0$ implies $c_n^2 < 1$. Furthermore $1 - c_n^2 \leq 1$. Hence

$$nc_n \leq 2\sqrt{a}, \qquad \Rightarrow \qquad c_n \leq \frac{2\sqrt{a}}{n} \to 0.$$

W. Van Assche, M. Foupouagnigni: *Analysis of non-linear recurrence relations for the recurrence coefficients of generalized Charlier polynomials*, J. nonlinear Math. Phys. **10**, supplement 2 (2003), 231–237

discrete Painlevé equations

$$\text{d-P}_{\text{I}} \qquad x_{n+1} + x_n + x_{n-1} = \frac{z_n + \gamma(-1)^n}{x_n} + \delta$$

$$\text{d-P}_{\text{II}} \qquad x_{n+1} + x_{n-1} = \frac{x_n z_n + \gamma}{1 - x_n^2}$$

$$\text{d-P}_{\text{IV}} \qquad (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - \kappa^2)(x_n^2 - \mu^2)}{(x_n + z_n)^2 - \gamma^2}$$

$$\text{d-P}_{\text{V}} \qquad \frac{(x_{n+1} + x_n - z_{n+1} - z_n)(x_n + x_{n-1} - z_n - z_{n-1})}{(x_{n+1} + x_n)(x_n + x_{n-1})}$$

$$= \frac{[(x_n - z_n)^2 - \alpha^2][(x_n - z_n)^2 - \beta^2]}{(x_n - \gamma^2)(x_n - \delta^2)}$$

with $z_n = \alpha n + \beta$ and $\alpha, \beta, \gamma, \delta, \kappa, \mu$ constants.

discrete Painlevé II ($\beta = \gamma = 0$, $\alpha = 1/\sqrt{a}$)

$$nc_n = \sqrt{a}(c_{n+1} + c_{n-1})(1 - c_n^2)$$

$$c_{n+1} = \frac{nc_n}{\sqrt{a}(1 - c_n^2)} - c_{n-1}$$

Singularities occur when $c_n = \pm 1$:

$$\begin{aligned} c_n &= \pm 1 \\ c_{n+1} &= \pm\infty \\ c_{n+2} &= \frac{\pm\infty}{-\infty} = ?? \end{aligned}$$

$$
\begin{aligned}
c_n &= 1+\epsilon \\
c_{n+1} &= -\frac{n}{2\sqrt{a}}\frac{1}{\epsilon} - \frac{n}{4\sqrt{a}} - c_{n-1} + \mathcal{O}(\epsilon) \\
c_{n+2} &= -1 + \frac{n+2}{n}\epsilon + \mathcal{O}(\epsilon^2) \\
c_{n+3} &= \frac{n+1}{\sqrt{a}(n+2)} - \frac{n}{n+2}\ c_{n-1} + \mathcal{O}(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
c_n &= -1+\epsilon \\
c_{n+1} &= -\frac{n}{2\sqrt{a}}\frac{1}{\epsilon} + \frac{n}{4\sqrt{a}} - c_{n-1} + \mathcal{O}(\epsilon) \\
c_{n+2} &= 1 + \frac{n+2}{n}\epsilon + \mathcal{O}(\epsilon^2) \\
c_{n+3} &= -\frac{n+1}{\sqrt{a}(n+2)} - \frac{n}{n+2}\ c_{n-1} + \mathcal{O}(\epsilon)
\end{aligned}
$$

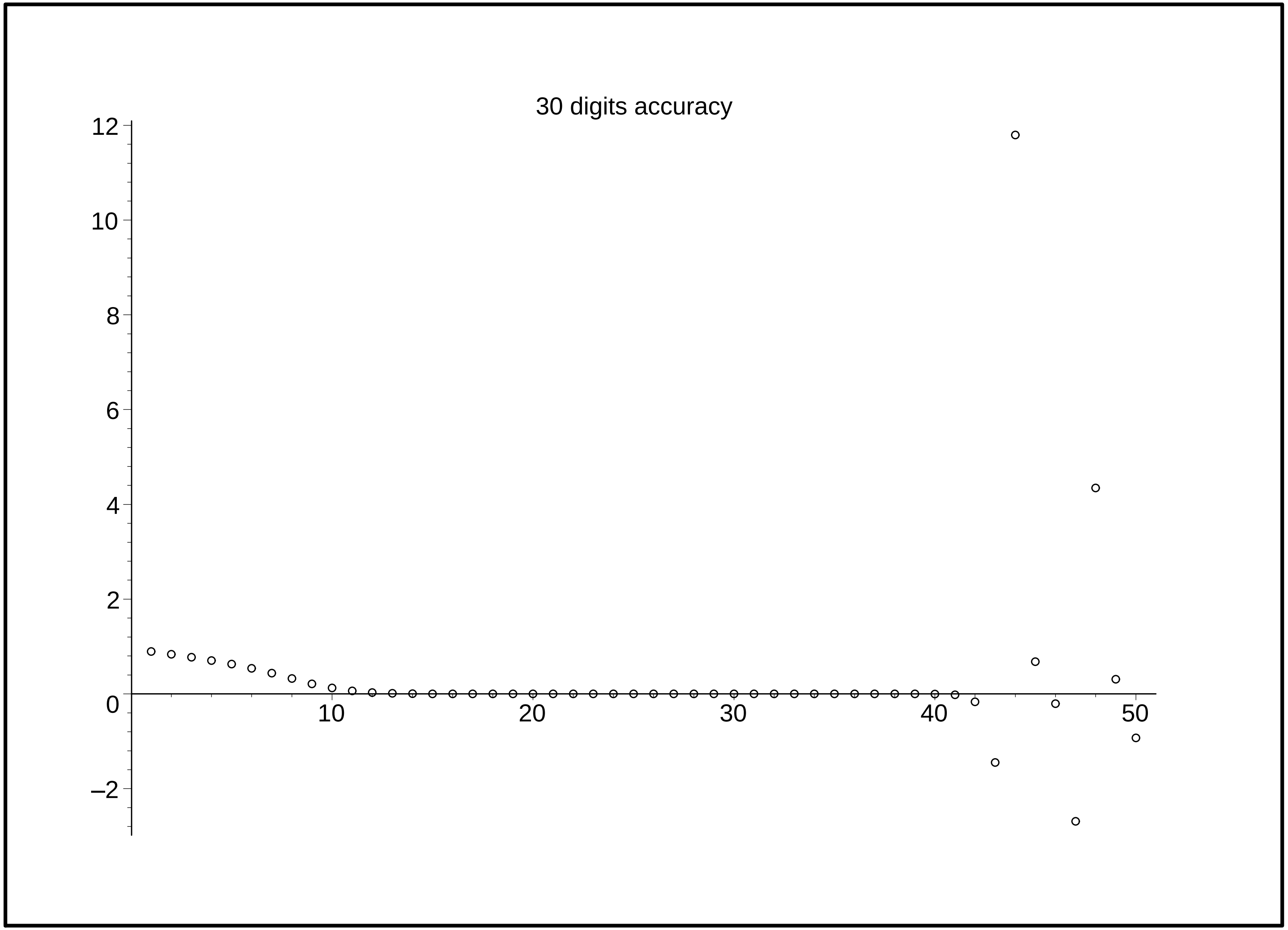
Singularity confined to c_{n+1}, c_{n+2}.

$$nc_n = \sqrt{a}(c_{n+1} + c_{n-1})(1 - c_n^2)$$

The recurrence relation is very unstable for computing the recurrence coefficients. Initial condition:

$$c_0 = 1, \quad c_1 = \frac{I_1(2\sqrt{a})}{I_0(2\sqrt{a})}$$

A small error in c_1 eventually gives $|c_n| > 1$ or $a_n < 0$.



30 digits accuracy
12
10
8
6
4
2
0
–2
10
20
30
40
50

Theorem 3 *Let*

$$nc_n = \sqrt{a}(c_{n+1} + c_{n-1})(1 - c_n^2)$$

Then there exists a unique solution with $c_0 = 1$ *and* $0 < c_n < 1$ *for* $n \geq 1$.

Proof: Operator $T = (x_0, x_1, x_2, \ldots) \mapsto (y_0, y_1, y_2, \ldots)$

$$\begin{aligned} y_0 &= 1 \\ y_n &= \frac{2\sqrt{a}(x_{n+1} + x_{n-1})}{n + \sqrt{n^2 + 4a(x_{n-1} + x_{n+1})^2}}, \qquad n \geq 1 \end{aligned}$$

Sequences with $x_0 = 1$ and $0 < x_n < 1 \mapsto$ sequences with $y_0 = 1$ and $0 < y_n < 1$.

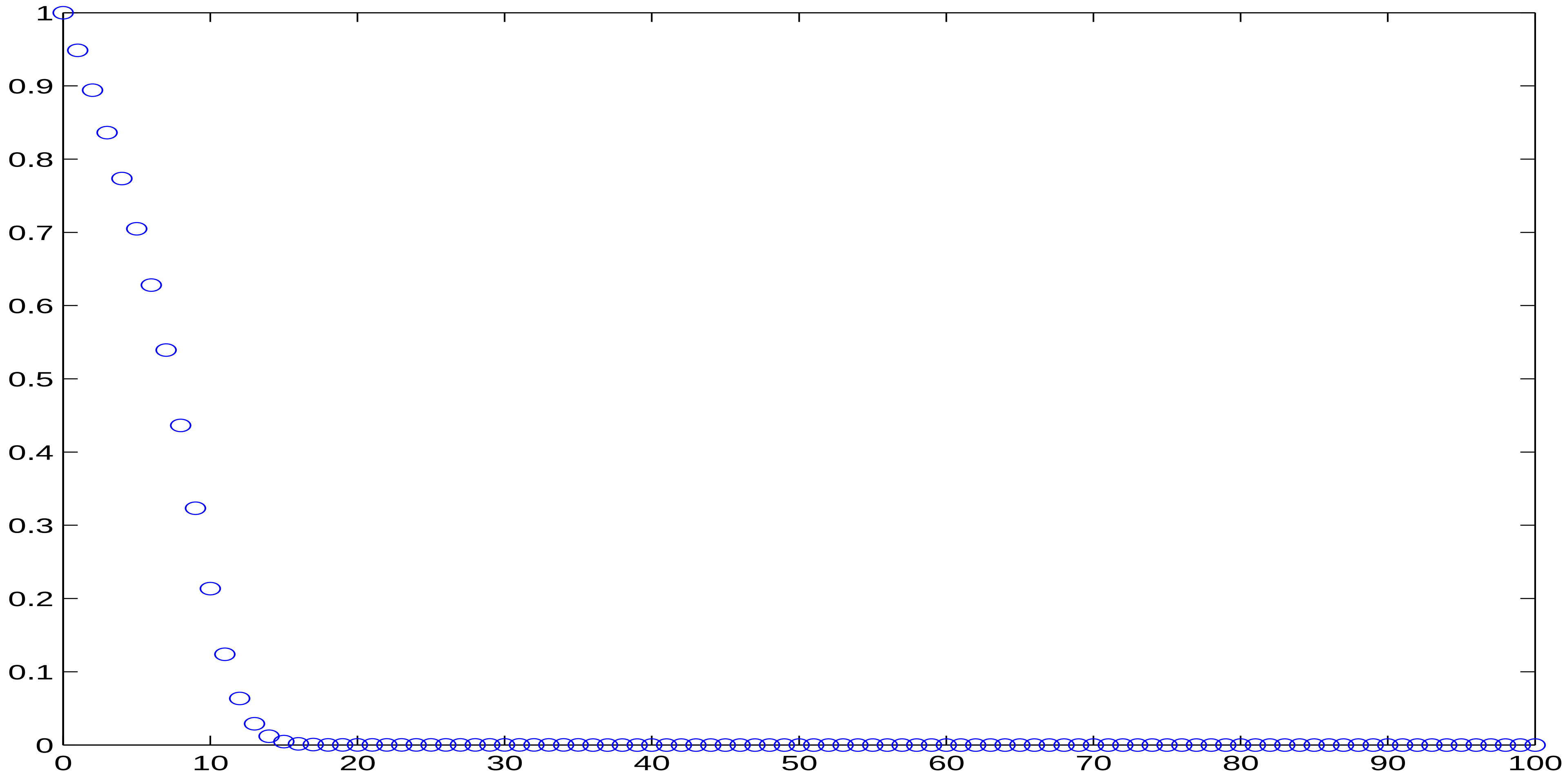
T^k is a contraction for some $k \Rightarrow T$ has a unique fixed point $Tc = c$.

This fixed point satisfies the non-linear recurrence relation.

A stable way to compute the recurrence coefficients:

computing $(c_1, c_2, \ldots, c_n)$
N iterations

```
x=[1,ones(1,n+N)];
for k=1:N
 y(1)=1;
 for j=2:n+N-k+1
  y(j)=2*sqrt(a)*(x(j+1)+x(j-1))/(j-1+sqrt((j-1)^2+4*a*(x(j+1)+x(j-1))^2));
 end
 x=y(1:n+N-k+1);
end
```


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80
90
100

semiclassical Laguerre

Consider the weight

$$w(x) = x^{\alpha} e^{-x^2+tx}, \qquad x \in [0, \infty)$$

with orthogonal polynomials satisfying

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

The Pearson equation is

$$[x w(x)]' = (-2x^2 + tx + \alpha + 1) w(x)$$

Put $y_n = 2a_n^2 - n - \alpha/2$ and $x_n = (t - 2b_n)/\sqrt{2}$ then

$$\begin{cases} x_n x_{n-1} = \frac{y_n^2 - \alpha^2/4}{y_n + z_n} \\ y_n + y_{n+1} = x_n(t/\sqrt{2} - x_n) \end{cases}$$

with $z_n = n + \alpha/2$.

$$\begin{cases} x_n x_{n-1} = \frac{y_n^2 - \alpha^2/4}{y_n + z_n} \\ y_n + y_{n+1} = x_n(t/\sqrt{2} - x_n) \end{cases}$$

This is a limiting case of asymmetric Painlevé IV

$$\begin{aligned} u_n u_{n-1} &= \frac{a(v_n + z_n - b)}{v_n^2 - \gamma^2} \\ v_n + v_{n+1} &= \frac{c}{u_n} + \frac{z_{n+1/2} + d}{u_n - 1} \end{aligned}$$

$u_n = 1/\epsilon x_n$, $v_n = \epsilon y_n$, $z_n = \epsilon n + \epsilon\alpha/2$, $a = 1/\epsilon$, $b = 0$, $c = 1/\epsilon + t/\sqrt{2}$, $d = -1/\epsilon$, $\gamma = \epsilon\alpha/2$ and $\epsilon \to 0$.

If we symmetrize the weight to

$$w(x) = |x|^{2\alpha+1} e^{-x^4+tx^2}, \qquad x \in (-\infty, \infty)$$

then the recurrence coefficients A_n of the orthogonal polynomials for this weight satisfy discrete Painlevé I

$$4A_n^2(A_{n-1}^2 + A_n^2 + A_{n+1}^2 - t/2) = n + (2\alpha + 1)\Delta_n$$

The recurrence coefficients for the weight on $[0, \infty)$ and the symmetrized weight on $(-\infty, \infty)$ are connected by

$$a_n = A_{2n}A_{2n-1}, \qquad b_n = A_{2n}^2 + A_{2n+1}^2$$

This gives a Miura transformation relating $\mathrm{dP_I}$ and asymmetric $\mathrm{dP_{IV}}$.

orthogonality on $(-\infty, \infty)$

The weight

$$w(x) = \begin{cases} c_1 x^\alpha e^{-x^2+tx}, & x > 0 \\ c_2 |x|^\alpha e^{-x^2+tx}, & x < 0 \end{cases}$$

satisfies the same Pearson equation

$$[xw(x)]' = (-2x^2 + tx + \alpha + 1)w(x)$$

The recurrence coefficients hence satisfy the same equations

$$\begin{cases} x_n x_{n-1} = \frac{y_n^2 - \alpha^2/4}{y_n + z_n} \\ y_n + y_{n+1} = x_n(t/\sqrt{2} - x_n) \end{cases}$$

with $a_0 = 0$ and $b_0(\beta)$ an expression depending on $\beta = c_1/c_2$.

generalized Hermite polynomials

When $c_1 = c_2$ and $t = 0$ then

$$w(x) = |x|^\alpha e^{-x^2}, \qquad x \in (-\infty, \infty)$$

then $b_n = 0$ and

$$\begin{cases} y_n^2 = \alpha^2/4 \\ y_n + y_{n+1} = 0. \end{cases}$$

so that we get a special solution $2a_n^2 = n + \alpha\Delta_n$ (Chihara)

$$\Delta_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

special case: $t = 0$

For the weight

$$w(x) = x^{\alpha} e^{-x^2}, \qquad x \in [0, \infty),$$

the recurrence coefficients a_n obey

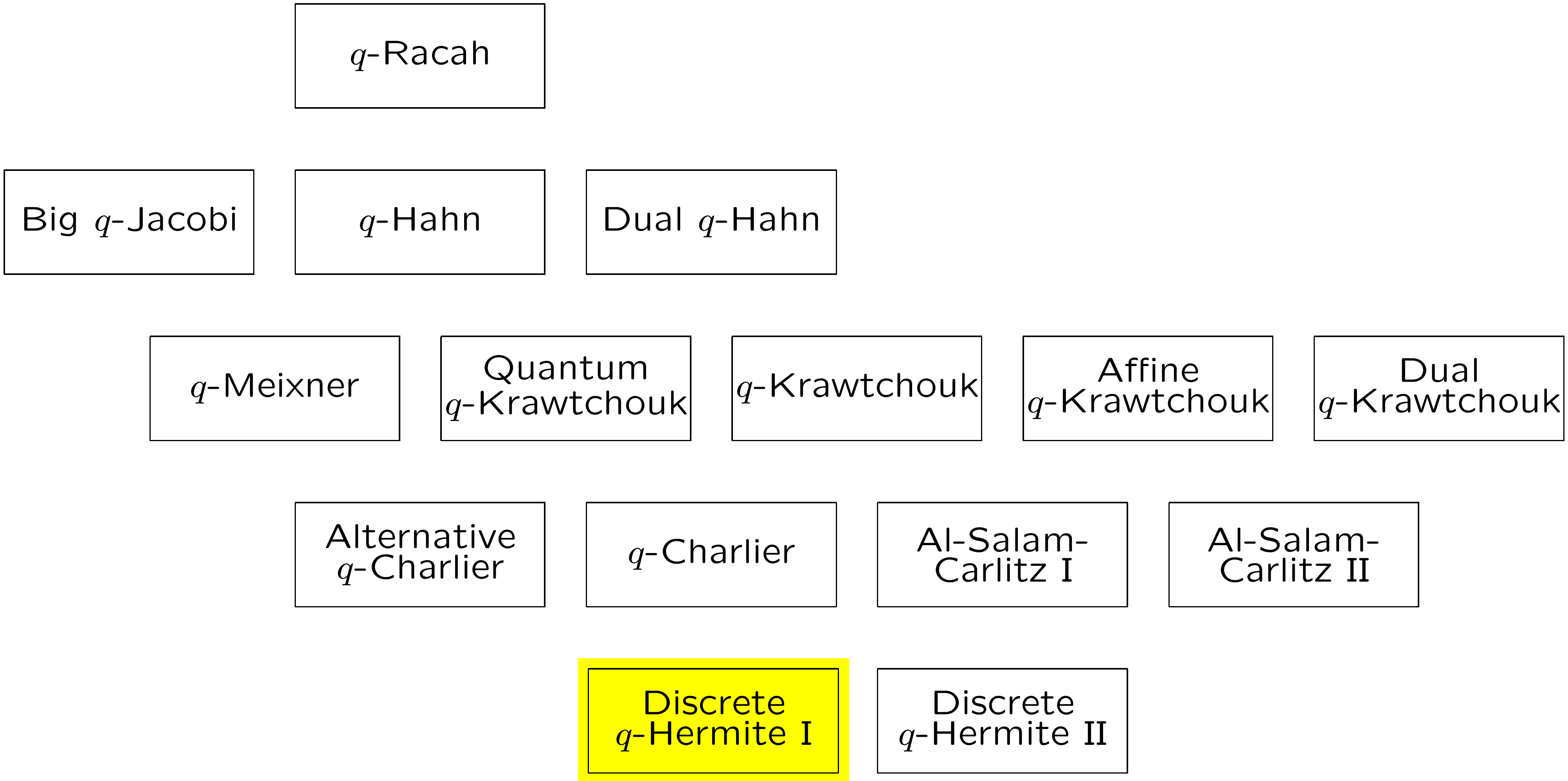
$$(y_n + y_{n+1})(y_n + y_{n-1}) = \frac{(y_n^2 - \alpha^2/4)^2}{(y_n + z_n)^2}$$

where $y_n = 2a_n^2 - n - \alpha/2$, $z_n = n + \alpha/2$. The recurrence coefficients b_n can be obtained from

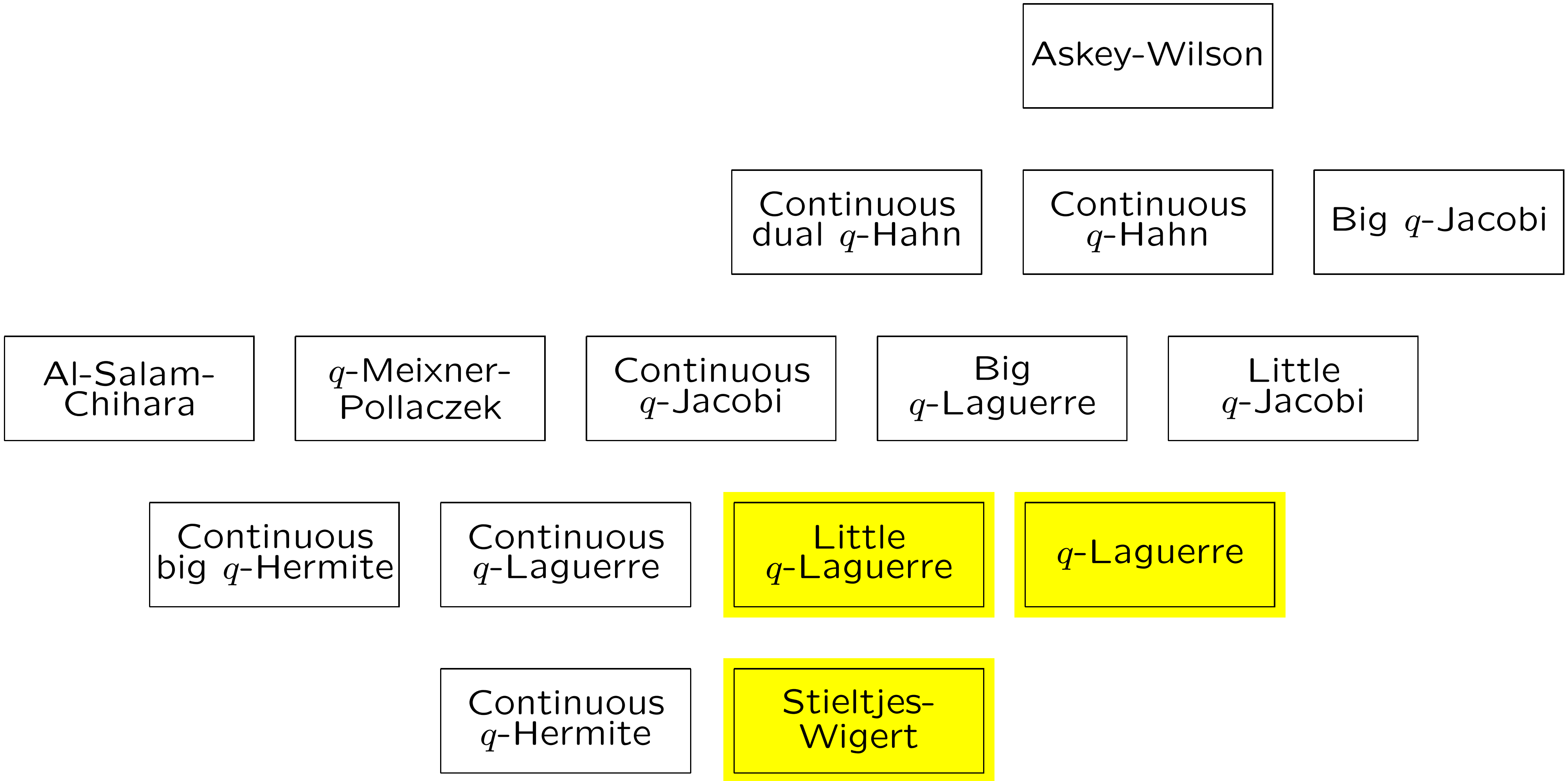
$$b_n^2 = n + \frac{\alpha + 1}{2} - a_n^2 - a_{n+1}^2.$$

This gives discrete Painlevé IV.

q-Askey tableau: right



q-Askey tableau: left



q-discrete orthogonal polynomials

$$\int_{-1}^{1} p_n(x)p_m(x)w(x)\,d_qx = \delta_{m,n}$$

with q-integral

$$\int_0^1 f(x)\,d_qx = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k, \qquad 0 < q < 1.$$

$w(x) = (q^2x^2;q^2)_\infty$ gives q-discrete Hermite I polynomials:

Pearson equation

$$(1-x^2)w(x) = w(x/q)$$

Recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_np_{n-1}(x), \qquad a_n^2 = q^{n-1}(1-q^n)$$

The weight $w(x) = (q^4x^4; q^4)_\infty$ is a q-deformation of $\exp(-x^4)$: q-discrete Freud polynomials.

Pearson equation

$$(1 - x^4)w(x) = w(x/q)$$

and structure relation

$$D_q p_n(x) = \frac{p_n(x) - p_n(qx)}{x(1-q)} = A_n p_{n-1}(x) + B_n p_{n-3}(x)$$

The recurrence coefficients satisfy $a_n^2 = q^{n-1}y_n$ with

$$1 - y_n^2 = q^n(y_{n+1}y_n + 1)(y_{n-1}y_n + 1)$$

(q-discrete Painlevé I equation)

F.W. Nijhoff: *On a q-deformation of the discrete Painlevé I equation and q-orthogonal polynomials*, Lett. Math. Phys. **30** (1994), 327–336

Another q-deformation is $w(x) = (x^2q^2; q^2)_\infty (cx^2q^2; q^2)_\infty$ with $c \leq 1$. The Pearson equation is

$$(1 - x^2)(1 - cx^2)w(x) = w(x/q)$$

and the structure relation is

$$D_q p_n(x) = A_n p_{n-1}(x) + B_n p_{n-3}(x)$$

The compatibility between the recurrence relation and the structure relation gives $a_n^2 = q^{n-1} y_n$ with

$$(1 - y_n)(1 - cy_n) = q^n (cy_{n+1}y_n - 1)(cy_{n-1}y_n - 1)$$

q-discrete Painlevé I equation ?

$$\int_{-1}^{1} p_n(x)p_m(x)w(x)\, d_qx = \delta_{m,n}$$

Use the weight*

$$w(x) = |x|^\alpha (x^2q^2; q^2)_\infty (cx^2q^2; q^2)_\infty$$

Then $y_n = a_n^2 q^{1-n}$ satisfies

$$q^{n-\alpha}(-cy_ny_{n+1} + q^\alpha)(-cy_ny_{n-1} + q^\alpha) = \begin{cases} (q^\alpha - y_n)(q^\alpha - cy_n)q^{-\alpha}, & n \text{ even} \\ (1 - y_n)(1 - cy_n), & n \text{ odd} \end{cases}$$

*L. Boelen, C. Smet, W. Van Assche: *q-Discrete Painlevé equations for recurrence coefficients of modified q-Freud orthogonal polynomials*, J. Difference Eq. Appl.

Put $u_n = y_{2n} q^{-\alpha}$ and $v_n = c y_{2n+1}$, then

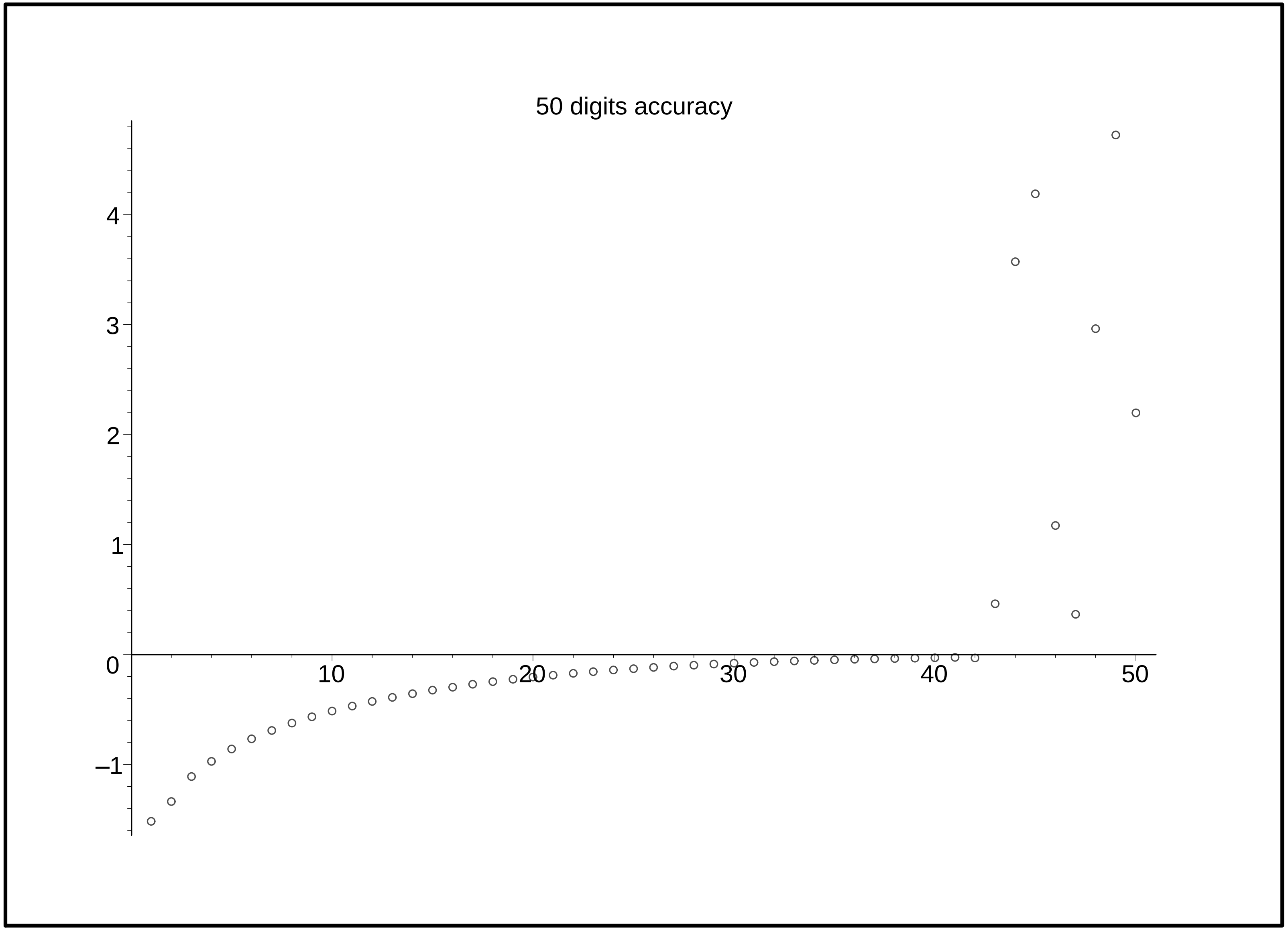
$$\begin{cases} q^{2n}(1-u_n v_n)(1-u_n v_{n-1}) = (1-u_n)(1-c u_n) \\ q^{2n+\alpha+1}(1-u_n v_n)(1-u_{n+1} v_n) = (1-v_n)(1-v_n/c) \end{cases}$$

This a limiting case of asymmetric q-$\mathrm{P_V}$ [α-q-$\mathrm{P_V}$ or E_6^q]

$$(1-u_n v_n)(1-u_n v_{n-1}) = \frac{(u_n - 1/p)(u_n - 1/r)(u_n - 1/s)(u_n - 1/t)}{(u_n - b\rho_n)(u_n - \rho_n/b)}$$

$$(1-u_n v_n)(1-u_{n+1} v_n) = \frac{(v_n - p)(v_n - r)(v_n - s)(v_n - t)}{(v_n - a\omega_n)(v_n - \omega_n/a)}$$

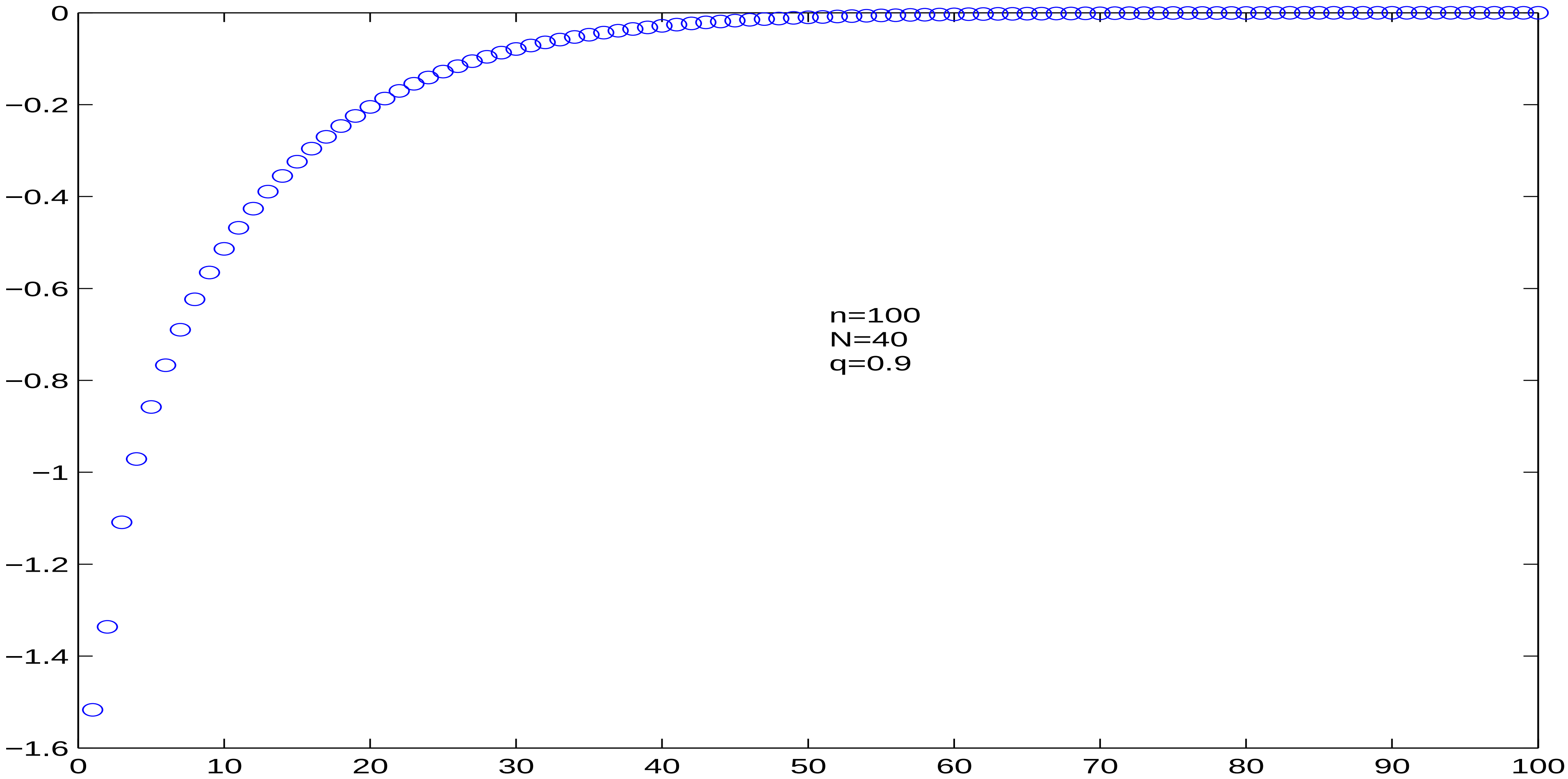
$p = 1$, $r = c$, $s = \kappa$, $t = 1/c\kappa$, $b = c\kappa$, $\rho_n = q^{2n}$, $a = \kappa$, $\omega_n = q^{2n+\alpha+1}$ and $\kappa \to 0$.


50 digits accuracy
4
3
2
1
0
–1
10
20
30
40
50

Theorem 4 *Let*

$$(1 - y_n)(1 - cy_n) = q^n(cy_{n+1}y_n - 1)(cy_{n-1}y_n - 1)$$

with $c \leq 1$. *Then there exists a unique solution with* $y_0 = 0$ *and* $0 < y_n < 1$ *for* $n \geq 1$. *Furthermore, for this solution one has* $\lim_{n\to\infty} y_n = 1$.


n=100
N=40
q=0.9
0
−0.2
−0.4
−0.6
−0.8
−1
−1.2
−1.4
−1.6
0
10
20
30
40
50
60
70
80
90
100

q-Laguerre polynomials

The q-Laguerre polynomials have weight function

$$w(x) = \frac{x^\alpha}{(-x;q)_\infty}, \qquad x \in [0,\infty).$$

$$\int_0^\infty p_n(x)p_m(x)\frac{x^\alpha}{(-x;q)_\infty}\,dx = \delta_{m,n}.$$

The recurrence coefficients are

$$\begin{cases} a_n^2 = q^{-4n-2\alpha+1}(1-q^n)(1-q^{n+\alpha}) \\ b_n = q - -2n - \alpha - 1[1 - q^n + q(1-q^{n+\alpha})] \end{cases}$$

A semiclassical extension is

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty}, \qquad x \in [0,\infty).$$

semiclassical q-Laguerre polynomials

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)_\infty}, \qquad x \in [0,\infty).$$

$$\int_0^\infty p_n(x)p_m(x)\frac{x^\alpha}{(-x^2;q^2)_\infty}\,dx = \delta_{m,n}.$$

if we put $x_n = q^{-n-\alpha/2}(1-q^{2n+\alpha-1}a_n^2)$, then

$$(x_n x_{n+1}-1)(x_n x_{n-1}-1) = \frac{(x_n - q^{-\alpha/2})^2(x_n-q^{\alpha/2})^2}{q^{2n+\alpha}(x_n-q^{-n-\alpha/2})^2}$$

which is q-Painlevé V

b_n is obtained from

$$b_n^2 x_n^2 q^{2n+\alpha} = 1+x_n x_{n+1}+(x_n x_{n-1}-1)(1-q^{n+\alpha/2}x_n)^2+2x_n(x_n-q^{\alpha/2}-q^{-\alpha/2}).$$

little q-Laguerre

Orthogonality on $\{q^n,\ n = 0, 1, 2, \ldots\}$ with weight $w(x) = x^\alpha (qx; q)_\infty$

$$\int_0^1 p_n(x) p_m(x) x^\alpha (qx; q)_\infty \, d_q x = \delta_{m,n}$$

Related to q-Laguerre with $q \to 1/q$.

The recurrence coefficients are given by

$$\begin{cases} a_n^2 = q^{2n-1+\alpha}(1 - q^n)(1 - q^{n+\alpha}) \\ b_n = q^n(1 - q^{n+\alpha+1}) + q^{n+\alpha}(1 - q^n) \end{cases}$$

A semiclassical extension is

$$w(x) = x^\alpha (q^2 x^2; q^2)_\infty, \qquad x \in \{q^n,\ n = 0, 1, 2, \ldots\}$$

semiclassical little q-Laguerre

$$w(x) = x^\alpha (q^2x^2;q^2)_\infty, \qquad x \in \{q^n,\ n = 0,1,2,\ldots\}$$

$$\int_0^1 p_n(x)p_m(x)x^\alpha(q^2x^2;q^2)_\infty\, d_qx = \delta_{m,n}$$

Put $a_n^2 = q^{2n+\alpha-1}(x_nq^{-n-\alpha/2}-1)$, then

$$(x_nx_{n-1}-1)(x_nx_{n+1}-1) = \frac{q^{2n+\alpha}(x_n-q^{\alpha/2})^2(x_n-q^{-\alpha/2})^2}{(x_n-q^{n+\alpha/2})^2}$$

which is again q-discrete Painlevé V

b_n is obtained from

$$b_n^2x_n^2q^{-2n-\alpha} = 1-x_nx_{n+1}-q^{2n}(x_nx_{n-1}-1)(x_nq^{-\alpha/2}-q^n)^2+2(x_n-q^{\alpha/2})(x_n-q^{-\alpha/2})$$

Stieltjes-Wigert polynomials

Consider the weight function $w(x) = \exp(-\gamma^2 \log^2 x)$ on $[0, \infty)$, then the orthogonal polynomials have recurrence coefficients

$$\begin{cases} a_n^2 = q^{-4n+1}(1 - q^n) \\ b_n = q^{-2n-1}[1 + q - q^{n+1}] \end{cases} \qquad q = \exp(-1/2\gamma^2).$$

This is an indeterminate moment problem. There are many more measures with the same moments, for example

$$w(x) = \frac{x^\alpha}{(-x;q)_\infty(-q/x;q)_\infty}, \qquad x \in [0, \infty)$$

with $\alpha = 1/2$.

semiclassical Stieltjes-Wigert polynomials

Consider the weight

$$w(x) = \frac{x^\alpha}{(-x^2;q^2)(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

Then for $x_n = q^{n-1}a_n^2 - q^{-n-\alpha}$ we have

$$x_{n-1}x_{n+1} = \frac{(x_n + q^{-\alpha})^2}{(q^{n+\alpha}x_n + 1)^2}$$

which is q-discrete Painlevé III.

The coefficient b_n is obtained from

$$q^{2n+\alpha}b_n^2 x_n = x_{n+1} + q^{2n+2\alpha}x_{n-1}(x_n + q^{-n-\alpha})^2 + 2(x_n + q^{-\alpha}).$$

generalized Stieltjes-Wigert

The weight function is

$$w(x) = \frac{x^{\alpha}(-p/x;q)_{\infty}}{(-x;q)_{\infty}(-q/x;q)_{\infty}}, \qquad x \in [0,\infty)$$

$p = 0$ gives the Stieltjes-Wigert polynomials

$p = q$ gives the q-Laguerre polynomials

A semiclassical variation is

$$w(x) = \frac{x^\alpha(-p^2/x^2;q^2)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \qquad x \in [0,\infty)$$

Then $a_n^2 = pq^{-n-\alpha/2}x_n + q^{-2n-\alpha+1}$ with

$$(x_nx_{n-1}-1)(x_nx_{n+1}-1) = \frac{(x_n+q^{1-\alpha/2}/p)^2(x_n+pq^{\alpha/2-1})^2}{(pq^{n+\alpha/2-1}x_n+1)^2}$$

which is q-discrete Painlevé V.

The coefficient b_n can be found from

$$q^{2n+\alpha}b_n^2x_n^2 = x_nx_{n+1} - 1 + q^{2n+2\alpha}(pq^{-1-\alpha/2}x_n + q^{-n-\alpha})^2(x_nx_{n-1}-1) + 2(x_n+q^{-\alpha/2+1}/p)(x_n+pq^{-1+\alpha/2}).$$