On Stability of the Metric Projection Operator

András Kroó

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest

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Joint work with Allan Pinkus

Outline of the talk

1. **Metric projections, distance between subspaces**
2. **Stability of metric projections in Hilbert spaces**
3. **Strong uniqueness in Best Approximation**
4. **Stability of metric projection via strong uniqueness**
5. **Stability of metric projection in smooth Banach Spaces**
6. **Open Problems**

1. Metric projections, distance between subspaces

Let X be a normed linear space and consider a subset M of X. For a given $f \in X$ denote by $P_M f$ the set of best approximations to f from M. Thus

$$P_M f := \{m^* : m^* \in M, \|f - m^*\| = \inf_{m \in M} \|f - m\|\}.$$

P_M is said to be the **metric projection** operator onto M. In general, $P_M f$ is a set-valued mapping. In this talk we always assume that $P_M f$ is non-empty, and also that $P_M f$ is single-valued, i.e., we have the uniqueness of the best approximation.

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A natural question that arises with respect to P_M is its **stability**. The stability of $P_M f$ with respect to *small perturbations of* f, that is continuity properties of the metric projection, have been widely investigated in the literature. In this talk we are interested in a different problem, namely:

How stable is the metric projection P_M relative to small perturbations of the closed linear subspace M?

In order to address this question we need a measure of **distance between subspaces**. A measure that will be convenient for our purposes is:

$$d(M,N) := \max\left\{ \sup_{\substack{m\in M\\ \|m\|=1}} \inf_{n\in N} \|m-n\|, \sup_{\substack{n\in N\\ \|n\|=1}} \inf_{m\in M} \|n-m\| \right\}. \tag{1.1}$$

This measure is symmetric in M and N, equals 0 if and only if $M = N$, and is a number between 0 and 1. This measure was introduced by Krein, Krasnolselski in the Hilbert space setting, and was later extended to Banach spaces by Krein, Krasnolselski, Milman.

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The following properties of $d(M,N)$ can be found in the literature when M and N are closed linear subspaces of a Banach space X:

1. If $\dim M > \dim N$, then $d(M,N) = 1$.

2. If $M^\perp$ denotes the annihilator of M in the dual space X^* $M^\perp := \{f : f \in X^*, f(m) = 0 \text{ all } m \in M\}$ then $d(M,N) = d(M^\perp, N^\perp)$.

3. If $X = H$ is a Hilbert space let $\{m_1, ..., m_r\}$ and $\{n_1, ..., n_r\}$ be orthonormal bases for M and N, respectively, with r finite. Let G denote the $r \times r$ matrix $G = ((m_i, n_j))_{i,j=1}^r$. Then $d(M,N) = [1-\lambda_r^2]^{1/2}$, where λ_r^2 is the smallest eigenvalue of GG^*.

2. Stability of metric projections in Hilbert spaces

Our goal: **Estimate $\|P_M f - P_N f\|$ in terms of $d(M,N)$.**

Typically such an estimate will be of order $d(M,N)^\beta$ with some $\beta \le 1$ which, in general, depends on the geometry of the space X.

Let us say that the metric projection is stable of order $\beta \le 1$ at M if

$$\|P_M f - P_N f\| \le c_{M,f} d(M,N)^\beta, \quad \forall f \in X, \forall N.$$

The main model will be $X = L^p$ for $p \in [1,\infty]$.

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When $X = H$ is a **Hilbert space**, then this theory is well understood. A proof of the fact that when $X = H$ is a Hilbert space

$$\|P_M - P_N\| = d(M, N)$$

was given by Akhiezer and Glazman. Here

$$d(M, N) = [1 - \lambda_r^2]^{1/2},$$

where λ_r^2 is the smallest eigenvalue of GG^*.

Stability of the metric projection is not always present.

Example. For $t \in (0,1)$, let M_t and N_t be 1-dimensional subspaces of $\mathbf{R^2}$ spanned by $(1,t)$ and $(1,-t)$, respectively. The space $\mathbf{R^2}$ is endowed with the uniform norm. Then $d(M_t,N_t) = 2t/(1+t)$. Let $f = (0,1)$. Direct computation shows that there is no stability because

$$P_{M_t}f = \left(\frac{1}{1+t}, \frac{t}{1+t}\right), \; P_{N_t}f = \left(-\frac{1}{1+t}, \frac{t}{1+t}\right),$$

$$\|P_{M_t}f - P_{N_t}f\|_\infty = \frac{2}{1+t} = \frac{d(M_t,N_t)}{t}.$$

3. Strong uniqueness in Best Approximation

Strong uniqueness of best approximations has been extensively investigated over the past 40 years, see the recent survey
Kroó, Pinkus: Strong Uniqueness, Surveys in Approximation Theory , 2010

Assume, as previously, that M is a closed linear subspace of a normed linear space X, and P_M is its corresponding metric projection (which, in general is not a linear operator).

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Definition. *The metric projection P_M is said to be strongly unique of order $\alpha > 0$ at M if for each $f \in X$ and every $m \in M$ we have*

$$\gamma_M(f)\|P_M f - m\|^\alpha \leq \|f - m\|^\alpha - \|f - P_M f\|^\alpha \qquad (*)$$

with some constant $\gamma_M(f) > 0$ depending only on f and M.

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Note: if $\|f - m\| \sim \|f - P_M f\| + \epsilon$ then $(*)$ implies $\|P_M f - m\| \sim \epsilon^{\frac{1}{\alpha}}$.

$\alpha =$ **order of strong uniqueness**.

We always have $\alpha \geq 1$ and $\gamma_M(f) \leq 1$. When $X = H$ is a Hilbert space then

$$\|m - P_M f\|^2 = \|f - m\|^2 - \|f - P_M f\|^2,$$

for all $m \in M$, implying that $\alpha = 2$ and $\gamma_M(f) = 1$.

What about the order of strong uniqueness in L^p?

Let $X = L^p$, $1 < p \le 2$, M be a closed linear subspace therein. Then for any $m \in M$

$$(p-1)\|P_M f - m\|_p^2 \le \|f - m\|_p^2 - \|f - P_M f\|_p^2.$$

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If $X = L^p$, $2 < p < \infty$, then for any $m \in M$

$$c_p\|P_M f - m\|_p^p \leq \|f - m\|_p^p - \|f - P_M f\|_p^p, \quad c_p \sim 2^{2-p}.$$

This gives the orders of L^p strong uniqueness, namely
$\alpha = 2$ for $1 < p < 2$, and $\alpha = p$ when $2 < p < \infty$,
and the constants $\gamma_M(f)$ can be chosen **independent of both f and M**.

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The case $p = \infty$ is quite different. In **real** $C(K)$, K a compact Hausdorff space, $M \subset C(K)$ a **Haar subspace**, metric projections satisfy strong uniqueness of **optimal order** $\alpha = 1$**.**(Newman, Shapiro)
Haar subspace: elements have $\le \dim M - 1$ zeros in K.
However the constant $\gamma_M(f)$ depends upon f and M, and is not uniformly bounded from below away from zero for all f in the unit ball of $C(K)$. Thus we have optimal order 1, but we have to pay a price in complications due to $\gamma_M(f)$. In **complex** $C(K)$, it is known that strong uniqueness of **order** $\alpha = 2$ **holds**.

In L^1 order of strong uniqueness in general may depend on f.

4. Stability of the Metric Projection and Strong Uniqueness

It turns out that we can estimate the stability of the metric projection under the assumption of strong uniqueness.

Let M be a closed linear subspace of the normed linear space X such that the metric projection P_M satisfies the strong uniqueness condition

$$\gamma_M(f)\|P_M f - m\|^\alpha \le \|f - m\|^\alpha - \|f - P_M f\|^\alpha,\ m \in M \qquad (*)$$

Then for any $f \in X$ and any closed linear subspace $N \subset X$

$$\|P_M f - P_N f\| \le \frac{10}{\gamma_M(f)^{1/\alpha}} d(M, N)^{1/\alpha}.$$

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Good news: unified approach to stability of metric projection in different spaces!

Bad news: This not always yields sharp estimates for stability of metric projection...

Strong uniqueness of order α implies stability of order $\frac{1}{\alpha}$.

For instance in a Hilbert Space we have strong uniqueness of order $\alpha = 2$ but we also know that $\|P_M - P_N\| = d(M, N)$, that is stability is of order 1!

So here is the good part:

Let M be a finite-dimensional subspace in $C(K)$ with P_M being a single valued metric projection. This requires that M be a Haar space. Then for any closed linear subspace N in $C(K)$ and any $f \in C(K)$

$$\|P_M f - P_N f\|_\infty \leq c_{M,f} d(M, N)^\beta,$$

where $\beta = 1$ in the real case, $\beta = 1/2$ in the complex case, and $c_{M,f}$ is a constant depending on f and M.

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Thus in **real** $C(K)$ **we obtain optimal order 1 stability** of metric projection!

It should be noted that we do not require that the second subspace N has a single valued metric projection, P_N can be set-valued and above holds with any element of best approximation from $P_N f$. Thus, as a byproduct, we obtain an upper bound for the diameter, diam $P_N f$, of the set $P_N f$ via the distance between M and N:

For any subspace N in $C(K)$ and any $f \in C(K)$ we have $\operatorname{diam} P_N f \leq 2c_{M,f} d(M,N)^\beta$ with $\beta = 1$ in the real case, $\beta = 1/2$ in the complex case.

Stability of metric projections in L^p:

Let M and N be closed linear subspace of L^p, $1 < p < \infty$.

Then for $f \in L^p, \|f\|_p \leq 1$ and $2 < p < \infty$,

$$\|P_M - P_N\|_p \leq c_p d(M,N)^{1/p},$$

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How about the sharpness of these orders of stability?

Given subspaces M and N with bases $m_1, ..., m_r$ and $n_1, ..., n_r$, respectively denote by $M + tN$ the subspace spanned by $m_1 + tn_1, ..., m_r + tn_r, t \in \mathbf{R}$.

Assume that M is any finite dimensional subspace in $L^p, 2 < p < \infty$ which possesses a nontrivial element **vanishing on a set of positive measure**. Then for any space N of the same dimension

$$\|P_M - P_{M+tN}\|_p \geq c|t|^{\frac{1}{p-1}} = cd(M, M+tN)^{\frac{1}{p-1}}.$$

Thus when $p > 2$ we have a lower bound which is "close" to the upper bound given above (but not quite matching it...) Unfortunately, no lower bounds for $p < 2$...

Here is an example showing that **in L^1 the stability of the metric projection can be of arbitrary order** depending on f.

Example. In $L^1[-1,1]$ let M be the set of constant functions, N the 1-dimensional space spanned by $g(x) = (x+1)/t, x \in [-1, -1+t]$ and $g(x) = 1, x \in [-1+t, 1]$, t is any fixed value in $(0,1)$. $d(M,N)$ can be easily calculated since 1 is the best approximation to g from M, while g is the best approximation to 1 from N. In fact $d(M,N) = t/(4-t)$. Consider the functions $f(x) = |x|^a \text{sgn} x$, for $a \in (0,1)$. Clearly $P_M f = 0$. Also $P_N f = (t/4)^a g$ is the best approximation to f from N. This yields

$$||P_M f - P_N f||_1 \geq \frac{3}{8} t^a.$$

On the other hand we clearly have $d(M,N) \leq t$. This means that the quantity $||P_M f - P_N f||_1$ can not be bounded from above by $c_{M,f} d(M,N)^\beta$ with a β independent of a (i.e. independent of f). This is very different from the situation in L^p, $p > 1$, or $C(K)$, where such a β always exists.

5. Stability of metric projection in smooth Banach Spaces

With a different approach, a better stability estimate for the metric projection in L^p, $p > 2$, can be exhibited. More explicitly, under some conditions on the **finite-dimensional** subspace M, for every subspace N of the same dimension $\|P_M f - P_N f\|$ is of order $d(M, N)$. (Just like in the Hilbert Space situation!) This is a considerable improvement when compared with $d(M, N)^{1/p}$. This refinement fundamentally uses that for $p \in (2, \infty)$ the L^p norm is twice differentiable. Can be extended to smooth Banach Spaces with this property.

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We first give a definition of **Gateaux differentiability** of P_M **with respect to the subspace** M.

Let M be an n dimensional linear subspace of X. The metric projection P_M is Gatoux differentiable at M if for any $f \in X \setminus M$, any basis $\{m_i, 1 \leq i \leq n\}$ in M and any subset $N = \{n_i, 1 \leq i \leq n\} \subset X$, for the space $M + tN := \{m_i + tn_i, 1 \leq i \leq n\}$ the limit

$$\lim_{t \to 0} \frac{P_{M+tN} f - P_M f}{t}$$

exists.

We assume that μ is a positive measure on a set K and $L^p(K, \mu)$ is the standard L^p-space.

Definition. Assume M is a subspace of $L^p(K, \mu)$. We say that M satisfies the Z_μ property if $\mu\{x : m(x) = 0\} = 0$ for every $m \in M$, $m \neq 0$.

Note that if $\dim M > 1$, then this implies that μ must be a non-atomic measure.

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The next statement gives a characterization of Gateaux differentiability of metric projections in $L^p(K, \mu)$.

Let M be an r-dimensional subspace of $L^p(K, \mu)$, $p \geq 2$, where μ is a non-atomic measure. If $p = 2$, then P_M is Gateaux differentiable. If $p > 2$, then P_M is Gateaux differentiable at M if and only if M satisfies the Z_μ property.

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The Gateaux derivative of the metric projection $P_M f$ **with respect to** f (not M!) was studied by Holmes and Kripke, who showed that it holds in L^p, $p > 2$ for every f, M. Hence the Gateaux differentiability of metric projections with respect to underlying subspaces is somewhat more restrictive...

Gateaux differentiability of metric projection in L^p, $p > 2$ helps to establish that if the Z_μ property holds for a finite-dimensional subspace M in $L^p(K, \mu)$, $p > 2$, then the size of $\|P_M f - P_N f\|_p$ is of order at most $d(M, N)$. Moreover the Z_μ property is necessary to attain this improvement.

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Theorem. Consider the space $L^p(K, \mu)$, $p > 2$, where μ is a non-atomic measure. Let M be an r-dimensional subspace of $L^p(K, \mu)$ satisfying the Z_μ property, i.e. $\mu\{x : m(x) = 0\} = 0$ for every $m \in M \setminus \{0\}$. Then for every $f \in L^p(K, \mu)$, $p > 2$, there exists a constant $c_{M,f}$ depending on M and f such that for any r-dimensional subspace N of $L^p(K, \mu)$, $p > 2$, we have

$$\|P_M f - P_N f\|_p \leq c_{M,f} d(M, N).$$

Furthermore, the Z_μ property of M is necessary in order for the above estimate to hold.

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Furthermore, the Z_μ property of M is necessary in order for the above estimate to hold.

This provides **optimal order 1 of stability** of metric projection in $L^p(K, \mu)$, $p > 2$!

Earlier we have shown a weaker stability of order $\frac{1}{p}$. Of course this improvement holds only for finite dimensional M satisfying the Z_μ property, and the constant above depends on f, M.

How does the twice differentiability of the norm in $L^p(K,\mu)$, $p > 2$ come into the play?

Let $f \in L^p(K,\mu) \setminus M$, $p \geq 2$, where M is an r-dimensional subspace of $L^p(K,\mu)$ satisfying the Z_μ property spanned by $m_1, ..., m_r$. For any subspace G spanned by $g_1, ..., g_r$ let $P_{M+tG} f = \sum a_i(t)(m_i + tg_i)$ denote the best $L^p(K,\mu)$ approximation to f from $M + tG$. Then for any fixed t the function

$$\min_{c_j} \int_K |f - \sum_{1 \leq j \leq r} c_j(m_j + tg_j)|^p d\mu$$

attains minimum when $c_j = a_j$, i.e., setting $\frac{\partial}{\partial c_j} = 0$ yields for every t and $1 \leq i \leq r$

$$S_i := \int_K |f - \sum_{j=1}^r a_j(m_j + tg_j)|^{p-1} sgn(f - \sum_{j=1}^r a_j(m_j + tg_j))(m_i + tg_i)\, d\mu = 0.$$

So we have a nonlinear system of r equations with respect to unknowns $a_1(t), ..., a_r(t)$. Since $p \geq 2$ we can differentiate again with respect to a_j yielding for $t = 0$

$$\frac{\partial S_i}{\partial a_j} = (1-p) \int_K |f - P_M f|^{p-2} m_i m_j d\mu,\ i, j = 1, ..., r.$$

Thus we have the nonlinear system

$$S_i(a_1(t), ..., a_r(t)) = 0, \quad 1 \leq i \leq r$$

with partial derivatives at $t = 0$ given by

$$\frac{\partial S_i}{\partial a_j} = (1-p)\int_K |f - P_M f|^{p-2} m_i m_j d\mu, \; i, j = 1, ..., r.$$

and Jacobian

$$J(0) = \det\left(\frac{\partial S_i}{\partial a_j}\right)_{i,j=1}^{r}$$

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In order to verify the differentiability of coefficient functions $a_j(t)$ we need to show that $J(0) \neq 0$. Indeed, if $J(0) = 0$ then

$$\sum_{i,j=1}^{r} b_i b_j \int_K |f - P_M f|^{p-2} m_i m_j d\mu = 0$$

for some $(b_1, ..., b_r) \neq 0$. But this then implies that $\int_K |f - P_M f|^{p-2} m^2 d\mu = 0$ where $m = \sum_{i=1}^{r} b_i m_i \in M \setminus \{0\}$. If $p = 2$ then we actually get $\int_K m^2 \, d\mu = 0$ which is an immediate contradiction. For $p > 2$ we see that this implies that m vanishes μ-a.e. on the set where $f - P_M f$ does not vanish. Since $f \in L^p(K, \mu) \setminus M$ this contradicts our assumption that M satisfies the Z_μ property. Hence $J(0) \neq 0$.

Thus we have a nonlinear system for coefficient functions $a_j(t)$ of

$$P_{M+tG}f = \sum a_i(t)(m_i + tg_i)$$

which has a non vanishing Jacobian at 0 provided that M satisfies the Z_μ property. This verifies differentiability of $a_j(t)$ which in turn yields the Gateaux differentiability of metric projection with respect to underlying subspace. Furthermore, with some extra work the derivatives of the coefficient functions can be shown to be uniformly bounded in the neighborhood of the origin and this implies Lip 1 property, or equivalently the **order 1 stability of the metric projection**.

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This is a brief outline of the sufficiency of the Z_μ property for order 1 stability of the metric projection in $L^p(K, \mu)$, $p > 2$.

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This is a brief outline of the sufficiency of the Z_μ property for order 1 stability of the metric projection in $L^p(K, \mu)$, $p > 2$.

The necessity part follows from a statement that was mentioned earlier in the talk:

If M is any finite dimensional subspace in $L^p, 2 < p < \infty$ for which the Z_μ property fails then for any space N of the same dimension

$$\|P_M - P_{M+tN}\|_p \geq ct^{\frac{1}{p-1}} = cd(M, M+tN)^{\frac{1}{p-1}}.$$

This lower bound shows that order 1 stability (or Gateaux differentiability) does not hold when the Z_μ property fails.

6. Some open problems

Clearly the Hilbert space situation $\|P_M - P_N\| = d(M, N)$ is the most satisfactory. However it is unclear how to calculate exactly $d(M, N)$ if both M and N are of infinite dimension and infinite co-dimension.

There are plenty of open problems in all other spaces.

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A. Open questions in $C(K)$.

In case of **real** $C(K)$ and a Haar subspace $M \subset C(K)$ we have order 1 stability

$$\|P_M(f) - P_N(f)\| \leq c_{f,M} d(M, N)$$

for any subspace N of the same dimension as M.

Can this estimate be made uniform in f, so that

$$\|P_M - P_N\| \leq c_{N,M} d(M, N)?$$

The answer is probably no, so a counter example could be a good way to start...

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When $C(K)$ is **complex** valued then a weaker order $\frac{1}{2}$ stability was shown above to hold for Haar spaces. Is this order of stability sharp? The answer is probably yes but a counter example is missing...

A. Open questions in L^p.

In L^p, $2 < p < \infty$, we have for any subspaces M, N uniform stability of order $\frac{1}{p}$:

$$\|P_M - P_N\| \leq c_p d(M, N)^{\frac{1}{p}}.$$

We also have seen that this order can not be better than $\frac{1}{p-1}$. This lower bound is not that much off but it does not quite match the upper bound....

So what is the correct order of stability of the metric projection in L^p, $2 < p < \infty$?

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So what is the correct order of stability of the metric projection in L^p, $2 < p < \infty$?

It was shown above that in L^p, $2 < p < \infty$ linear stability

$$\|P_M(f) - P_N(f)\| \leq c_{f,M} d(M,N)$$

onto finite-dimensional subspaces holds if and only if the subspace M satisfies the Z_μ property.

Is the condition of finite dimensionality necessary here, as well?

Furthermore, assuming the Z_μ property, then the constant $c_{M,f}$ exhibits a dependence on f.

It would be of interest to determine whether (or when) $c_{M,f} = c_M \|f\|_p$?

In case $1 < p < 2$ we have order order of stability $\frac{1}{2}$ but no lower bound is known...