Thin knotted vortex tubes in stationary solutions to the Euler equation

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Definitions

Let's consider the Euler equation in $\mathbb{R}^3$, which describes an ideal fluid:

$$\partial_t u + (u \cdot \nabla) u = -\nabla P, \qquad \operatorname{div} u = 0.$$

The unknowns are:

- The velocity of the fluid, $u(x,t)$ (a time-dependent vector field in $\mathbb{R}^3$).
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In this talk we will consider the stationary case, in which the velocity does not depend on time. Physically, this situation is generally associated with equilibrium configurations.

In what follows, we will only consider steady solutions to Euler, $u(x)$.

In the spirit of the Lagrangian approach to fluid mechanics, we will be interested in the existence of geometric structures in the body of the fluid. More precisely, these structures are defined via the trajectories of the most important vector field in fluid mechanics:

- Vorticity: $\omega := \operatorname{curl} u$.

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For our present purpose, the most important object in this talk is:

Definition (Vortex tube)

A vortex tube is a bounded domain $\mathcal{T} \subset \mathbb{R}^3$ such that $\partial\mathcal{T}$ is diffeomorphic to $\mathbb{T}^2$ and $\omega \parallel \partial\mathcal{T}$ (that is, $\partial\mathcal{T}$ is an invariant torus of the vorticity).

Setting up the problem

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Are there steady solutions to the Euler equation with knotted/linked vortex tubes?

(A knot is a (non-intersecting) closed curve in $\mathbb{R}^3$. A link is a union of disjoint knots.)

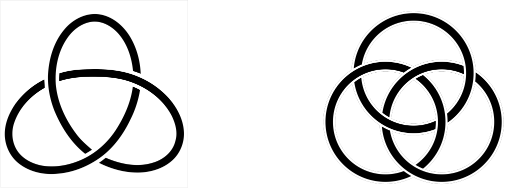

Figure : *Are there stationary vortex tubes given by a trefoil knot or by the Borromean rings?*

Our goal:

There are steady solutions to the Euler equation with thin vortex tubes "of arbitrarily complicated topology".

Motivation:

Lord Kelvin's conjecture (1875): knotted and linked thin vortex tubes can arise in steady solutions to the Euler equation.

"Thin" means that the width of the tube is small compared with its length.

- In time-dependent Euler, the vorticity is transported: $\partial_t \omega = [\omega, u]$, so

$$\omega(x,t) = (\phi_{t,t_0})_* \, \omega(x,t_0)\,,$$

with ϕ_{t,t_0} the non-autonomous flow of u. Hence vorticity evolves through a diffeomorphism. Kelvin's conjecture is motivated by this phenomenon (y by the belief in the existence of asymptotic states). A more sophisticated variant of this argument, which relies on non-rigurous reasoning using a MHD system, was put forward by Moffatt in the 1980s.

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- Physicists do use vortex tubes in their work, without arriving at inconsistencies. Indeed, Kelvin was also motivated by Maxwell's observations of "water twists" (although his own motivation was less down-to-earth). Observed in the lab and used in engineering.
- Linked vortex tubes have been used to construct possible blow-up scenarios for Euler*. Our statement does not have a direct bearing on this, although it would imply that the reason for a possible blow-up would certainly not be topological.

*Pelz, J. Fluid Mech. 2001

Beltrami fields

To prove our statement we will resort to a particular class of solutions to the Euler equation known as (strong) Beltrami fields:

$$\operatorname{curl} u = \lambda u\,, \qquad \lambda \text{ nonzero real constant.}$$

One can check that any such $u(x)$ solves Euler with pressure $P = -\frac{1}{2}|u|^2$.

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Why Beltrami?

- The Beltrami equation is linear (although not quite elliptic).
- Arnold's structure theorem.
  - **Theorem:** (Arnold 1966) Under mild hypotheses, if u and ω are not everywhere collinear, the vortex lines sit over a ver rigid "integrable" (a.k.a. "laminar") structure, roughly equivalent to linear flows on tori and cylinders, that makes it difficult to have complicated knotted vortex tubes or lines.

(vortex lines = trajectories of ω)

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- The Beltrami equation is linear (although not quite elliptic).
- Arnold's structure theorem.
- As conjectured by Arnold, in the case of vortex lines we do not have these problems with Beltrami fields:
  - **Theorem:** (E. & Peralta-Salas, Ann. of Math. 2012) *Given any finite link L, we can deform it with a small diffeomorphism Φ of $\mathbb{R}^3$ (close to the identity in C^k) so that $\Phi(L)$ is a set of vortex lines of a Beltrami field in $\mathbb{R}^3$.*

(vortex lines = trajectories of ω)

A digression: why Beltrami fields with constant factor?

As a matter of fact, Arnold's structure theorem only asserts that u should be a Beltrami field in the sense that

$$\operatorname{curl} u = f(x) u \,, \qquad \operatorname{div} u = 0 \,.$$

Intuitively speaking, the reason to consider only the case $f = \lambda$ (strong Beltramis) is that f is a first integral ($\nabla f \cdot u = 0$). Again, this forces the fluid flow to be "laminar".

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In fact, one can go well beyond this observation and prove that *for "most" functions f there are no nontrivial solutions to the equation, not even locally* (that is, in a small ball without imposing any boundary conditions). Therefore, we should not resort to general Beltramis to prove Kelvin's conjecture.

This is very closely related to (and explains) the helical flow paradox of Morgulis, Yudovich and Zaslavski (1994):

$$\text{Steady solution} \begin{cases} \text{Not Beltrami} \rightsquigarrow \text{laminar} \\ \text{Beltrami} \rightsquigarrow \begin{cases} f \neq \lambda \rightsquigarrow \text{laminar} \\ f = \lambda \rightsquigarrow \text{possibly turbulent, but "just a few"} \end{cases} \end{cases}$$

Statements about Beltrami fields with constant factor

Theorem (E & PS, 2014)

Suppose that u is a Beltrami field in $B \subset \mathbb{R}^3$ with factor $f \in C^{6,\alpha}(B)$. Then there is a sixth-order nonlinear partial differential operator $P \neq 0$, which can be computed explicitly, such that $u \equiv 0$ unless $P[f] \equiv 0$. In particular, $u \equiv 0$ for all f in an open and dense subset of C^k.

Corollary

Suppose that the vector field u is a Beltrami field with factor

$$f(x) := 1 + ax_1 + bx_1^3 + x_3$$

in a neighborhood of the origin. Then $u \equiv 0$ if $b \neq 0$.

Theorem (E & PS, 2014)

Suppose that the function f is of class $C^{2,\alpha}(B)$. If a regular level set $f^{-1}(c)$ has a connected component in B diffeomorphic to S^2, then $u \equiv 0$.

Key idea: which kind of equation do we have?

- Elliptic-like: Since a Beltrami field satisfies

$$-\Delta u = f^2 u + \nabla f \times u\,,$$

one has unique continuation and good regularity properties, and one can even prove Liouville theorems (Nadirashvili, GAFA 2014).

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- Constrained evolution: The equation for a Beltrami field turns out to be locally equivalent to the constrained evolution problem

$$\partial_t \beta = T\beta\,, \qquad d\beta = 0\,,$$

where β is a time-dependent 1-form on a surface and T is a time-dependent tensor field.

Back to Kelvin: describing thin tubes

A convenient way of constructing thin tubes of width ϵ is as the ϵ-tubular neighborhood of a closed curve γ in $\mathbb{R}^3$:

$$\mathcal{T}_\epsilon(\gamma) := \left\{x \in \mathbb{R}^3 : \mathrm{dist}(x, \gamma) < \epsilon\right\}$$

It is obvious any finite collection of disjoint tubes can be mapped into a set of thin tubes $\mathcal{T}_\epsilon(\gamma_1), \ldots, \mathcal{T}_\epsilon(\gamma_N)$ through a diffeomorphism of $\mathbb{R}^3$.

The realization theorem

Realization theorem (E. & Peralta-Salas, Acta Math., 2015)

Let $\gamma_1, \dots, \gamma_N$ be (nonintersecting, possibly knotted and linked) closed curves in $\mathbb{R}^3$. For all small enough ϵ and some nonzero λ, the collection of disjoint tubes of width ϵ $\mathcal{T}_\epsilon(\gamma_1), \dots, \mathcal{T}_\epsilon(\gamma_N)$ can be transformed using a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in the C^m norm, so that $\Phi[\mathcal{T}_\epsilon(\gamma_1)], \dots, \Phi[\mathcal{T}_\epsilon(\gamma_N)]$ are vortex tubes of a Beltrami field u, which satisfies $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ for some $\lambda \in \backslash\{0\}$ and falls off at infinity as $|D^j u(x)| < C_j/|x|$.

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- There is a positive-measure set of invariant tori of u (or ω) close to the boundary of each vortex tube. On each of these tori, the field is ergodic.
- In each vortex tube there are infinitely many closed vortex lines, including $\Phi(\gamma_i)$ (the core of each tube).

- The fact that the tubes are thin is crucial in the proof.
- A Beltrami field cannot be in $L^2(\mathbb{R}^3)$ because $\Delta u = -\lambda^2 u$, but the solutions we construct have optimal decay in the class of Beltrami fields.
- The vortex tubes we construct are stable:
  - They are Lyapunov-stable, that is, the vortex lines passing through a δ-neighborhood of each torus do not leave a δ'-neighborhood of the torus.
  - The invariant tori are preserved (up a small diffeomorphism) under suitably small perturbations of the field u.
- As a corollary of the fact that there is a vortex line diffeomorphic to each curve γ_i, we recover our previous result on the existence of vortex lines of any knot or link type.
- Since $v(x,t) := \mathrm{e}^{-\mu\lambda^2 t} u(x)$ is a solution to the Navier–Stokes equation

$$\partial_t v + (v \cdot \nabla) v = -\nabla P + \mu \Delta v \,, \qquad \operatorname{div} v = 0$$

in $\mathbb{R}^3$ with $P = -\frac{1}{2}|v|^2$, the theorem impies the existence of stationary knotted vortex tubes for Navier–Stokes (which appear in time-dependent solutions, however).

Strategy of proof

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The construction of steady solution to Euler with invariant tori is a problem with both PDE and topological aspects:

- Topological techniques are too "soft" to capture what happens inside a PDE[†].
- Purely analytical techniques have not been too successful so far in these matters, with the exception of partial results on steady axisymmetric solutions through variational methods[‡].

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Strategy

1. Construct a local Beltrami field v (that is, in a neighborhood of the tube) with a prescribed set of vortex tubes (or invariant tori).
2. Check that these invariant tori are robust (that is, they are preserved under small perturbations of the field v).
3. Approximate the local solution v by a global Beltrami field u (defined in all $\mathbb{R}^3$).

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Yes, yes, but what about finite energy solutions?

The same result holds true for high-energy Beltrami fields on the torus (or the sphere):

Theorem (E., Peralta-Salas and Torres de Lizaur, freshly baked)

Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes contained in a contractible subset of the flat torus $\mathbb{T}^3$. Then for any **large enough** odd integer λ there exists a Beltrami field u satisfying the equation $\operatorname{curl} u = \lambda u$ and a diffeomorphism Φ of $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.

"Large enough" implies, among many things, that new ideas are needed to prove this result!

Step 1: Construction of the local Beltrami field

Let's focus on a single closed curve γ and the corresponding tube $\mathcal{T}_\epsilon \equiv \mathcal{T}_\epsilon(\gamma)$.
Let's fix a harmonic field $h \neq 0$ in the tube tangent to $\partial\mathcal{T}_\epsilon$:

$$\operatorname{curl} h = 0 \quad \text{and} \quad \operatorname{div} h = 0 \quad \text{in } \mathcal{T}_\epsilon\,, \qquad h \parallel \partial\mathcal{T}_\epsilon\,.$$

By Hodge theory, h is unique up to a multiplicative constant and the L^2 projector onto harmonic fields in $\mathcal{P}v(x) := h(x)\,\|h\|_{L^2}^{-2}\int_{\mathcal{T}_\epsilon} h\cdot v$ (the harmonic part of v).

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"Local" existence theorem

If λ does not belong to a certain countable sets without accumulations, there is a unique solution to the problem

$$\operatorname{curl} v = \lambda v \quad \text{in } \mathcal{T}_\epsilon\,, \quad v \parallel \partial\mathcal{T}_\epsilon\,, \quad \mathcal{P}v = h\,.$$

Besides for small λ, v and h are close: $\|v - h\|_{H^k(\mathcal{T}_\epsilon)} \leqslant C_{\epsilon,k}|\lambda|\|h\|_{L^2(\mathcal{T}_\epsilon)}\,.$

- The fact that v and h are close is intuitively clear, but quite different from what happens in closed manifolds.
- We have chosen $v\cdot\nu = 0$, so $\partial\mathcal{T}_\epsilon$ is an invariant torus. But we cannot prescribe the tangential part of $v|_{\partial\mathcal{T}_\epsilon}$ (and that's important).

For the proof, we reduce the proof of the theorem to analyzing the auxiliary equation

$$\operatorname{curl} w = f \,,$$

in $\mathcal{T}_\epsilon$, where $f \in L^2$ is a divergence-free field and
$w \in \mathcal{W} := \{u \in H^1(\mathcal{T}_\epsilon) : u \parallel \partial\mathcal{T}_\epsilon\}$.

In turn, this auxiliary equation is equivalent to demanding

$$\int_{\mathcal{T}_\epsilon} \Big(\operatorname{curl} w \cdot \operatorname{curl} u + \operatorname{div} w \, \operatorname{div} u + \mathcal{P}w \cdot \mathcal{P}u \Big) = \int_{\mathcal{T}_\epsilon} f \cdot \operatorname{curl} u \quad \forall\, u \in \mathcal{W} \,.$$

The functional in the LHS is coercive (it's essentially the H^1 norm), so we can proceed using the Riesz representation theorem and Fredholm operators.

But. . .

Existence is not enough: the robustness of the invariant torus depends on KAM arguments, which require very fine information on the behavior of the local Beltrami field v in a neighborhood of $\partial\mathcal{T}_\epsilon$.

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Existence is not enough: the robustness of the invariant torus depends on KAM arguments, which require very fine information on the behavior of the local Beltrami field v in a neighborhood of $\partial\mathcal{T}_\epsilon$.

The only information we have is

$$\|v - h\|_{H^k(\mathcal{T}_\epsilon)} \leqslant C_{\epsilon,k}|\lambda|\,\|h\|_{L^2(\mathcal{T}_\epsilon)}\,,$$

so that for small λ it would be enough to analyze the harmonic field h. (We'll actually use a more refined version of this inequality, but let's skip it for now.)

Therefore, our next step in the construction of the local Beltrami field is a detailed analysis of Beltrami fields on $\mathcal{T}_\epsilon$ (tangent to the boundary).

Underlying idea: We are aiming for a fine result on thin tubes, but we haven't exploited the geometry of the tubes yet.

In order to analyze "the" harmonic field h in a thin tube, let's introduce coordinates adapted to the tube:

$$(\alpha, y) \in \mathbb{S}^1_\ell \times \mathbb{D}^2 \mapsto \gamma(\alpha) + \epsilon y_1 e_1(\alpha) + \epsilon y_2 e_2(\alpha) \in \mathcal{T}_\epsilon \,.$$

Here $\gamma(\alpha)$ is an arc-length parametrization of the curve γ and $e_1(\alpha), e_2(\alpha)$ are the normal and binormal vectors. We have set $\mathbb{S}^1_\ell := \mathbb{R}/\ell\mathbb{Z}$, with $\ell :=$ length of γ. We'll also use polar coordinates $y_1 = r\cos\theta$, $y_2 = r\sin\theta$, and denote by $\kappa(\alpha)$ y $\tau(\alpha)$ the curvature and torsion of the core curve of the tube, γ.

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With $B := 1 - \epsilon\kappa r\cos\theta$, one can check that the field

$$h_0 := B^{-2}\left(\partial_\alpha + \tau\,\partial_\theta\right)$$

is curl-free, so we can fix a harmonic field h by requiring

$$h = h_0 + \nabla\psi\,, \qquad \Delta\psi = -\operatorname{div} h_0\,, \qquad \partial_\nu\psi = 0\,.$$

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WARNING! $\quad \nabla\psi = \dfrac{\psi_\alpha + \tau\psi_\theta}{B^2}\,\partial_\alpha + \dfrac{\psi_r}{\epsilon^2}\,\partial_r + \dfrac{A\psi_\theta + \epsilon^2 r^2\tau\psi_\alpha}{(\epsilon r B)^2}\,\partial_\theta\,.$

Summing up: considering functions of $(\alpha, y) \in \mathbb{S}^1_\ell \times \mathbb{D}^2$ and using the notation

$$Q(\alpha, y) = \mathcal{O}(\epsilon^n) \iff \|Q\|_{H^k} \leqslant C_k \epsilon^n \quad \forall\, k\,,$$

our goal is to prove that the solution of the problem

$$\Delta\psi = -\operatorname{div} h_0 = \mathcal{O}(\epsilon) \quad \text{in } \mathbb{S}^1_\ell \times \mathbb{D}^2\,, \qquad \partial_\nu \psi = 0\,, \qquad \int_{\mathbb{S}^1_\ell \times \mathbb{D}^2} \psi = 0$$

satisfies:

- $\psi = \mathcal{O}(\epsilon^2)$
- $D_y \psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^4)$
- $\partial_\theta \psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^5)$

Both the actual expression of the functions that arise and the powers of ϵ appearing in the error terms are crucial in order to prove the robustness of the invariant tori through a KAM argument.

To control ψ we resort to L^2 estimates for the equation $\Delta\psi = \rho$ in $\mathbb{S}^1_\ell \times \mathbb{D}^2$ with Neumann boundary conditions $\partial_\nu \psi = 0$ and $\int_{\mathbb{S}^1_\ell \times \mathbb{D}^2} \psi = 0$. The guiding principle is

"We are rich in derivatives but poor in ϵ".

Recall that

$$\Delta\psi = \frac{1}{\epsilon^2}\Big(\psi_{rr} + \frac{1}{r}\psi_r + \frac{A}{r^2 B^2}\psi_{\theta\theta}\Big) + \frac{1}{B^2}\psi_{\alpha\alpha} + \frac{2\tau}{B^2}\psi_{\alpha\theta} + \frac{\tau' - \epsilon r(\kappa\tau' - \kappa'\tau)\cos\theta}{B^3}\psi_\theta$$
$$+ \frac{1}{\epsilon}\Big(\frac{\kappa\sin\theta(B^2 - (\epsilon\tau r)^2)}{rB^3}\psi_\theta - \frac{\kappa\cos\theta}{B}\psi_r\Big) + \frac{\epsilon r(\kappa'\cos\theta - \tau\kappa\sin\theta)}{B^3}\psi_\alpha\,.$$

Throughout, $\| \cdot \| \equiv \| \cdot \|_{L^2}$.

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- Besides, $\|\psi_y\| + \|\psi_{yy}\| \leqslant C\epsilon^2\|\rho\|$.
  This is better. The basic idea is to get rid of spurious dependence on α by taking $\bar{\psi}(\alpha, y) := \psi(\alpha, y) - \fint_{\mathbb{D}^2} \psi(\alpha, y')\, dy'$; basically, we gain that $\|\bar{\psi}\| \leqslant C\epsilon^2\|\rho\|$ and this affects the other estimates. Notice that $\partial_y^n \bar{\psi} = \partial_y^n \psi$.

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  This is better. The basic idea is to get rid of spurious dependence on α by taking $\bar{\psi}(\alpha, y) := \psi(\alpha, y) - \fint_{\mathbb{D}^2} \psi(\alpha, y')\, dy'$; basically, we gain that $\|\bar{\psi}\| \leqslant C\epsilon^2\|\rho\|$ and this affects the other estimates. Notice that $\partial_y^n \bar{\psi} = \partial_y^n \psi$.
- Besides, $\|\psi_{\alpha y}\| \leqslant C\epsilon^2\|\rho\|_{H^1}$ (optimal in ϵ).

Throughout, $\|\cdot\| \equiv \|\cdot\|_{L^2}$.

- $\|\psi\|_{H^2} \leqslant C_\epsilon \|\rho\|_{L^2}$ (almost useless: what is the dependence on ϵ?)
- $\|\psi\| + \|\psi_\alpha\| + \|\psi_{\alpha\alpha}\| + \frac{1}{\epsilon}\big(\|\psi_y\| + \|\psi_{\alpha y}\| + \|\psi_{yy}\|\big) \leqslant C\|\rho\|$.
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  This is better. The basic idea is to get rid of spurious dependence on α by taking $\bar{\psi}(\alpha, y) := \psi(\alpha, y) - ⨍_{\mathbb{D}^2} \psi(\alpha, y')\, dy'$; basically, we gain that $\|\bar{\psi}\| \leqslant C\epsilon^2 \|\rho\|$ and this affects the other estimates. Notice that $\partial_y^n \bar{\psi} = \partial_y^n \psi$.
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Why? Think of the simple equation $\psi_{\alpha\alpha} + \frac{\psi_{yy}}{\epsilon^2} = \rho$, with $\alpha, y \in \mathbb{S}^1$. With $\psi = \sum_{n,k} \widehat{\psi}_{nk}\, e^{in\alpha + iky}$, one has $\widehat{\psi}_{nk} = -\widehat{\rho}_{nk}/[n^2 + (\frac{k}{\epsilon})^2]$, so that

$$\|\psi_{\alpha y}\| = \left(\sum_{n,k} \frac{n^2 k^2}{[n^2 + (\frac{k}{\epsilon})^2]^2} |\widehat{\rho}_{nk}|^2 \right)^{\frac{1}{2}} \leqslant \begin{cases} \epsilon \Big(\sum_{n,k} \frac{n^2 + (\frac{k}{\epsilon})^2}{[n^2 + (\frac{k}{\epsilon})^2]^2} |\widehat{\rho}_{nk}|^2 \Big)^{\frac{1}{2}} \leqslant C\epsilon \|\rho\|\,, \\ \epsilon^2 \Big(\sum_{n,k} \frac{(\frac{k}{\epsilon})^2}{[n^2 + (\frac{k}{\epsilon})^2]^2} n^2 |\widehat{\rho}_{nk}|^2 \Big)^{\frac{1}{2}} \leqslant C\epsilon^2 \|\rho_y\|\,. \end{cases}$$

Theorem (Estimates for ψ)

For any $j \geqslant 1$ and $k \geqslant 0$ we have the bound

$$\|D_y^j \partial_\alpha^k \psi\|_{L^2} \leqslant C\epsilon^2 \|\rho\|_{H^{k+\max\{j-2,0\}}} .$$

Furthermore, $\|\partial_\alpha^k \psi\|_{L^2} \leqslant C_k \|\psi\|_{H^{\max\{k-2,0\}}}$.

How to control the local Beltrami field v

- Estimates for the Neumann BVP "optimal in ϵ", useful to extract information about the harmonic field h.
- Estimates for the curl equation "optimal in ϵ", useful to quantify how the local Beltrami field becomes close to h for small λ.
- Compute the leading terms in the equations (i.e., solve v, ψ up to errors of higher order in ϵ) and control the error using the estimates.

Step 2: Robustness of the invariant torus $\partial\mathcal{T}_\epsilon$

In Step 1, we have constructed a "local" Beltrami field v, which satisfies $\operatorname{curl} v = \lambda v$ in the tube $\mathcal{T}_\epsilon$ and is tangent to the boundary. Therefore, $\partial\mathcal{T}_\epsilon$ is an invariant torus of v. This is true for all small $\lambda \neq 0$.

Our next objective is to show that a field "close enough" to v also has an invariant torus "very close" to $\partial\mathcal{T}_\epsilon$. The natural tool for this is KAM theory, which we shall apply to the Poincaré map $\Pi : \mathbb{D}^2 \to \mathbb{D}^2$ of the local Beltrami field v in a transverse section of the tube.

Why use the Poincaré map Π?

Theorem (Preservation of area)

The Poincaré map Π preserves a measure $[1+\epsilon\kappa(0)\, r\cos\theta+\mathcal{O}(\epsilon^2)]\, r\, dr\, d\theta$ in the disk.

Theorem (computation of ω_Π and $\mathcal{N}_\Pi$)

For $\lambda=\mathcal{O}(\epsilon^3)$, the rotation number ω_Π and the normal torsion $\mathcal{N}_\Pi$ of Π are determined in terms of the geometry of the core curve as

$$\omega_\Pi=\int_0^\ell \tau(\alpha)\,d\alpha+\mathcal{O}(\epsilon^2)\,,$$

$$\mathcal{N}_\Pi=-\frac{5\pi\epsilon^2}{8}\int_0^\ell \kappa(\alpha)^2\,\tau(\alpha)\,d\alpha+\mathcal{O}(\epsilon^3)\,.$$

Computing these objects is hard and messy: one needs quantitative analysis of the trajectories of the field v which hinges on our previous analysis of harmonic fields on the tube. As we saw before, the requirements of this computation determine which estimates had to be developed for the PDEs.

Corollary

For a "generic" curve γ, ϵ small enough and $\lambda = \mathcal{O}(\epsilon^3)$,

1. The rotation number ω_Π is Diophantine.
2. The normal torsion $\mathcal{N}_\Pi$ is nonzero (non-degeneracy condition).

Why are we interested in computing these objects?

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Theorem (KAM for Beltrami fields, particular case)

For a "generic" curve γ, sufficiently small ϵ and $\lambda = \epsilon^3$, there exist k, δ' such that: if u is a Beltrami field in the whole $\mathbb{R}^3$ such that

$$\|u - v\|_{C^k(\mathcal{T}_\epsilon)} < \delta',$$

then u has an invariant torus given by $\Psi(\partial \mathcal{T}_\epsilon)$, where Ψ is a smooth diffeomorphism C^m-close to the identity. On this invariant torus, u is ergodic.

Note that this theorem does not guarantee the existence of the approximating global Beltrami field u!

Some remarks:

1. The Poincaré map is not Hamiltonian and there are no action-angle variables, so we're not in the usual KAM situation. Fortunately we can just invoke a KAM theorem in the literature[§]. The hypotheses are: Π analytic, measure-preserving, with Diophantine rotation number and nonzero normal torsion (that it, what we have striven to prove).
2. The reason why applying de la Llave's theorem is so hard is that the situation is highly degenerate: the field v is a small perturbation of the field

$$\partial_\alpha + \tau(\alpha)\,\partial_\theta$$

which has a constant rotation number. That's why we needed a very careful treatment of harmonic field on the tube. This is related to the fact that $\mathcal{N}_\Pi = \mathcal{O}(\epsilon^2)$, just as in the Newtonian three-body problem.
3. Herman's last geometric theorem easily yields the existence of a positive-measure set of invariant tori on which the field u is ergodic.
4. We would be done if we knew that there is a Beltrami field u in $\mathbb{R}^3$ approximating the local solution v in C^k!

[§] González-Enríquez & de la Llave, JDE 2008

Step 3: Approximation by a global Beltrami field

Applying the same reasoning to each tube $\mathcal{T}_\epsilon(\gamma_j)$, we obtain a vector field v that satisfies the equation curl $v = \lambda v$ in

$$S := \mathcal{T}_\epsilon(\gamma_1) \cup \cdots \cup \mathcal{T}_\epsilon(\gamma_N)\,,$$

with $\lambda \neq 0$. By the robustness result, if u is a Beltrami field in $\mathbb{R}^3$ and $\|u - v\|_{C^k(S)}$ is small enough, u has a collection of vortex tubes diffeomorphic to S.

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Theorem (Approximation)

There is a solution of the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ such that $\|u - v\|_{C^k(S)} < \delta$. Moreover, u falls off at infinity as $|D^j u(x)| < C_j/|x|$.

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What we need is a Runge-type theorem for curl. Problems to solve:

1. Curl is not elliptic, so Lax–Malgrange does not work.
2. We want solutions that decay at infinity, while in the classical Runge theorem we cannot have this property due to the Liouville theorem.
3. (By topological reasons, we cannot construct the local solutions using Cauchy–Kowalewski.)

How to prove this result:

1. First we construct a solution w of the Beltrami equation in a large ball B_R that contains all the tubes. This is done using Green's functions and duality arguments.
2. Once we have the solution w in B_R, we truncate its corresponding Fourier–Bessel series to obtain a global Beltrami field which falls off at infinity as $1/|x|$ and is of the form

$$u = \sum_{l=0}^{L} \sum_{m=-l}^{l} c_{lm} \, j_l(\lambda r) \, Y_{lm}(\theta, \varphi) \, ,$$

where (r, θ, φ), j_j and Y_{lm} are the spherical coordinates in $\mathbb{R}^3$, spherical Bessel functions and spherical harmonics, respectively.

Thank you for your attention!