The two matrix model with quartic potential

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Unitary ensembles

- **Probability measure on $n \times n$ Hermitian matrices**

$$\frac{1}{\tilde{Z}_n} e^{-n \operatorname{Tr} V(M)} \, dM$$

  - **This is GUE for** $V(M) = \frac{1}{2} M^2$

- **Explicit formula for joint density of eigenvalues**

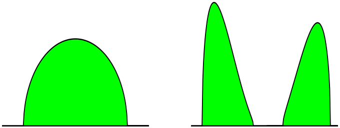
$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{n} e^{-nV(x_j)}$$

Global eigenvalue behavior

- **As $n \to \infty$, there is a limiting mean eigenvalue density $\rho_V(x)$.**
- **The probability measure** $\boxed{d\mu_V(x) = \rho_V(x)dx}$ **minimizes**

$$\iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int V(x)d\mu(x)$$

- **Typical behavior for polynomial V: density ρ_V is positive and real analytic on each interval and vanishes as a square root at endpoints.**



Orthogonal polynomials

- **Average characteristic polynomial**

$$P_{n,n}(x) = \mathbb{E}\left[\det(xI_n - M)\right]$$

is nth degree orthogonal polynomial with respect to $e^{-nV(x)}$ on real line

  - **Orthogonality with respect to varying weight**

- **Monic OPs** $\quad P_{k,n}(x) = x^k + \cdots$

$$\int_{-\infty}^{\infty} P_{k,n}(x)\, x^j\, e^{-nV(x)}\, dx = h_{k,n}\delta_{j,k}, \qquad j = 0, \ldots, k.$$

Determinantal correlation functions

- **Eigenvalues are determinantal point process with correlation kernel**

$$K_n(x,y) = \sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}} \sum_{k=0}^{n-1} \frac{P_{k,n}(x)P_{k,n}(y)}{h_{k,n}}$$

  - **This means that the k point eigenvalue correlation function (which is proportional to marginal density) is given by $k \times k$ determinant**

$$\det \left[K_n(x_i, x_j)\right]_{i,j=1}^{k}$$

- **Global eigenvalue behavior**

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \rho_V(x)$$

Local eigenvalue behavior

- **Local eigenvalue statistics have universal behavior as $n \to \infty$.**
  - **Sine kernel in the bulk: if $c = \rho_V(x^*) > 0$ then**

$$\lim_{n\to\infty} \frac{1}{cn} K_n\left(x^* + \frac{x}{cn}, x^* + \frac{y}{cn}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

Pastur, Shcherbina (1997), Bleher, Its (1999)

Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

McLaughlin, Miller (2008), Lubinsky (2009)

  - **Airy kernel at the spectral edge (if ρ_V vanishes as a square root at x^*)**

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_n\left(x^* + \frac{x}{cn^{2/3}}, x^* + \frac{y}{cn^{2/3}}\right) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

Singular behavior

- **Other limiting kernels at special points**
  - **Painlevé II kernels at interior points where density vanishes.**

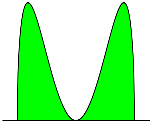
    Bleher, Its (2003), Claeys, K (2006)

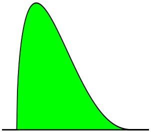
    Claeys, K, Vanlessen (2008), Shcherbina (2008)

  - **Painlevé I_2 kernels at edge points where density vanishes at higher order.**

    Claeys, Vanlessen (2007)

    Claeys, Its, Krasovsky (2010)





Riemann-Hilbert problem

- **Powerful tool for asymptotic analysis in case of real analytic V is the Riemann-Hilbert problem for OPs**

Fokas, Its, Kitaev (1992)

(1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic**

(2) Y **has limiting values** $Y_\pm$ **on** $\mathbb{R}$**, satisfying**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix} \quad \text{for } x \in \mathbb{R},$$

(3) $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$.

- **Correlation kernel is**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Steepest descent analysis

- **Asymptotics of orthogonal polynomials can be proved by means of a steepest descent analysis of the RH problem**
- **Essential role is played by minimizer of the energy functional**

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

and associated g-function

$$g(z) = \int \log(z-s) \rho_V(s) ds$$

that satisfies a number of (in)equalities due to Euler-Lagrange variational conditions associated with the minimization problem.

Long term project

- **Extend these results to other matrix ensembles where eigenvalues have determinantal structure**
  - **Random matrices with external source**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M)-AM)} dM$$

  - **Coupled random matrices (two matrix model)**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} dM_1 dM_2$$

  - **Normal matrix model (for complex matrices M)**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(MM^*-V(M)-V(M^*))} dM$$

- **We need extensions / analogues of**
  - **Orthogonal polynomials**
  - **Riemann-Hilbert problem**
  - **Equilibrium problem**

Two matrix model

- **The Hermitian two matrix model**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \, dM_1 dM_2$$

is a probability measure on pairs (M_1, M_2) of $n \times n$ Hermitian matrices.

- V **and** W **are polynomial potentials**
- $\tau \neq 0$ **is a coupling constant**

Biorthogonal polynomials

- **Average characteristic polynomials**

$$P_{n,n}(x) = \mathbb{E}\left[\det(xI_n - M_1)\right]$$

$$Q_{n,n}(y) = \mathbb{E}\left[\det(yI_n - M_2)\right]$$

are biorthogonal polynomials Mehta,Shukla (1994), Eynard, Mehta (1998)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_{n,n}(x)y^j e^{-n(V(x)+W(y)-\tau xy)}\,dxdy = 0,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j Q_{n,n}(y) e^{-n(V(x)+W(y)-\tau xy)}\,dxdy = 0,$$

for $j = 0, \ldots, n-1$.

Multiple orthogonality

- **Suppose W is a polynomial of degree $r+1$**
- **Then the biorthogonality condition for $P_{n,n}$ can be rewritten as MOP conditions with weights**

$$w_{j,n}(x) = e^{-nV(x)} \int_{-\infty}^{\infty} y^j e^{-n(W(y)-\tau xy)}\, dy, \qquad j = 0, \dots, r-1,$$

and multi-index (assume n is a multiple of r) $\vec{n} = (n/r, \dots, n/r)$

$$\int_{-\infty}^{\infty} P_{n,n}(x) x^k w_{j,n}(x)\, dx = 0,$$

for $k = 0, \dots, \frac{n}{r} - 1$ and $j = 0, \dots, r-1$.

K, McLaughlin (2005)

Riemann-Hilbert problem

- **Multiple orthogonality leads to RH problem of size $(r+1) \times (r+1)$.** Van Assche, Geronimo, K (2001)
- **In case** $\deg W = 4$, **(i.e.,** $r = 3$**)**

(1) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{4\times 4}$ **is analytic**

(2) Y **has limiting values** $Y_\pm$ **on** $\mathbb{R}$, **satisfying for** $x \in \mathbb{R}$

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) **as** $z \to \infty$

$$Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-n/3} & 0 & 0 \\ 0 & 0 & z^{-n/3} & 0 \\ 0 & 0 & 0 & z^{-n/3} \end{pmatrix}$$

Correlation kernel

- **RH problem has a unique solution and**

$$Y_{11}(z) = P_{n,n}(z)$$

- **Eigenvalues of M_1 are determinantal point process (multiple orthogonal polynomial ensemble) with correlation kernel**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & w_{2,n}(y) \end{pmatrix} \times Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- **This is based on the Christoffel-Darboux formula for multiple orthogonal polynomials.** Daems, K (2004)

Quartic potential

- **Two matrix model**

$$\frac{1}{Z_n} e^{-n\,\mathrm{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)}\, dM_1\, dM_2$$

with V even and W a quartic polynomial

$$W(y) = \frac{1}{4}y^4 + \frac{\alpha}{2}y^2$$

- **There is a vector equilibrium problem for three measures that describes the limiting mean density for the eigenvalues of M_1.**

Quartic potential

- **Two matrix model**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \, dM_1 \, dM_2$$

with V even and W a quartic polynomial

$$W(y) = \frac{1}{4} y^4 + \frac{\alpha}{2} y^2$$

- **There is a vector equilibrium problem for three measures that describes the limiting mean density for the eigenvalues of M_1.**

- **Earlier work for $\alpha = 0$ Duits, K (2009), Mo (2009)**
- **Very different description of limiting eigenvalue distributions is due to Guionnet (2004)**

Notation

$$I(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y)$$

Vector equilibrium problem

- **Energy functional** $E(\mu_1, \mu_2, \mu_3) =$

$$I(\mu_1) + I(\mu_2) + I(\mu_3) - I(\mu_1, \mu_2) - I(\mu_2, \mu_3) + \int V_1(x) d\mu_1(x) + \int V_3(x) d\mu_3(x)$$

- **Minimize** $E(\mu_1, \mu_2, \mu_3)$ **among** μ_1, μ_2, μ_3 **such that**

(a) μ_1 **is a measure on** $\mathbb{R}$ **with** $\mu_1(\mathbb{R}) = 1$**,**
(b) μ_2 **is a measure on** $i\mathbb{R}$ **with** $\mu_2(\mathbb{R}) = 2/3$**,**
(c) μ_3 **is a measure on** $\mathbb{R}$ **with** $\mu_3(\mathbb{R}) = 1/3$**, and**

$$\mu_2 \leq \sigma_2,$$

where σ_2 **is certain given measure on** $i\mathbb{R}$

External field V_1 acting on first measure

Definition

$$V_1(x) = V(x) + \min_{s\in\mathbb{R}}(W(s) - \tau xs)$$

$$W(s) = \tfrac{1}{4}s^4 + \tfrac{\alpha}{2}s^2$$

- **By Laplace's method**

$$V_1(x) = -\lim_{n\to\infty}\frac{1}{n}\log\int_{-\infty}^{\infty} e^{-n(V(x)+W(s)-\tau xs)}ds$$

- $W(s) - \tau xs$ **has a minimum for** $s \in \mathbb{R}$ **at** $s_1(x)$

$$V_1(x) = V(x) + W(s_1(x)) - \tau x s_1(x)$$

External field V_3 acting on the third measure

- $W(s) - \tau x s$ **has more local extrema on the real line, say at** $s_2(x)$ **and** $s_3(x)$**, if** $|x| < x^*(\alpha)$ **where**

$$x^*(\alpha) = \frac{2}{\tau}\left(\frac{-\alpha}{3}\right)^{3/2} \text{ if } \alpha < 0, \qquad x^*(\alpha) = 0 \text{ if } \alpha \geq 0,$$

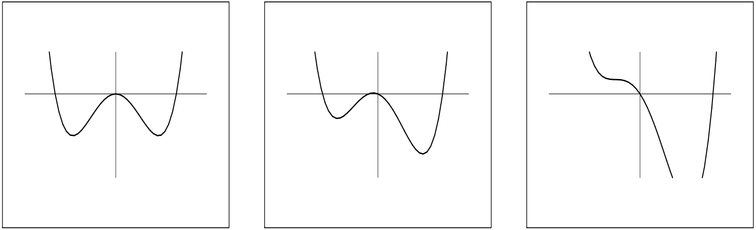
Definition

if $|x| < x^*(\alpha)$ then

$$V_3(x) = \underbrace{(W(s_3(x)) - \tau x s_3(x))}_{\text{local maximum}} - \underbrace{(W(s_2(x)) - \tau x s_2(x))}_{\text{other local minimum}} > 0$$

if $|x| \geq x^*(\alpha)$ then $V_3(x) = 0$

Illustration



- For x close to 0, the function $W(s) - \tau s x$ has a global minimum and one local non-global minimum and one local maximum. Then $V_3(x)$ is the difference between the value at the local non-global local minimum and the local maximum.
- For large x, there is only a global minimum and then $V_3(x) = 0$.

Upper constraint σ_2 acting on the second measure

- $W(s) - \tau z s$ **with** $z \in i\mathbb{R}$ **has a global minimum on the imaginary axis**
- **If** $z = iy$ **with** $|y| > y^*(\alpha) = \begin{cases} \frac{2}{\tau}\left(\frac{\alpha}{3}\right)^{3/2}, & \alpha > 0, \\ 0 & \alpha \leq 0, \end{cases}$

  then there no other local extrema

Definition

σ_2 is the measure on $(-i\infty, -iy^*(\alpha)] \cup [iy^*(\alpha), i\infty)$ with density

$$\frac{d\sigma_2(z)}{|dz|} = \frac{\tau}{\pi} \operatorname{Re} s(z)$$

where $s(z)$ is the solution of $W'(s) = \tau z$ with $\operatorname{Re} s(z) > 0$.

Vector equilibrium problem

- **Minimize**

$$I(\mu_1) + I(\mu_2) + I(\mu_3) - I(\mu_1, \mu_2) - I(\mu_2, \mu_3) + \int V_1(x) d\mu_1(x) + \int V_3(x) d\mu_3(x)$$

among measures μ_1, μ_2, μ_3 **such that**

(a) μ_1 **is a measure on** $\mathbb{R}$ **with** $\mu_1(\mathbb{R}) = 1$**,**

(b) μ_2 **is a measure on** $i\mathbb{R}$ **with** $\mu_2(\mathbb{R}) = 2/3$**,**

(c) μ_3 **is a measure on** $\mathbb{R}$ **with** $\mu_3(\mathbb{R}) = 1/3$**, and**

$$\mu_2 \leq \sigma_2$$

Results on equilibrium measures

Theorem (Duits, K, Mo)

There is a unique minimizer (μ_1, μ_2, μ_3) *of the vector equilibrium problem.*

Structure of minimizer

- **The support of μ_1 is a finite union of intervals**

$$S(\mu_1) = \bigcup_{j=1}^{N} [a_j, b_j]$$

- **The support of μ_2 is equal to the support of σ_2, and the constraint σ_2 is active on a symmetric interval around 0, possibly empty,**

$$S(\sigma_2 - \mu_2) = i\mathbb{R} \setminus (-ic_2, ic_2), \qquad c_2 \geq 0.$$

- **The support of μ_3 has at most one gap**

$$S(\mu_3) = \mathbb{R} \setminus (-c_3, c_3), \qquad c_3 \geq 0$$

Riemann surface

- **The supports of the measures**

$$S(\mu_1), \quad S(\sigma_2 - \mu_2), \quad S(\mu_3)$$

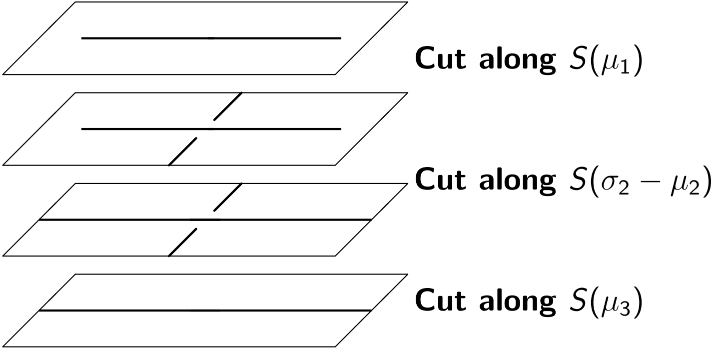
are the cuts for a four-sheeted Riemann surface

- **Four sheets**

$$\begin{aligned}
\mathcal{R}_1 &= \overline{\mathbb{C}} \setminus S(\mu_1) \\
\mathcal{R}_2 &= \mathbb{C} \setminus (S(\mu_1) \cup S(\sigma_2 - \mu_2)) \\
\mathcal{R}_3 &= \mathbb{C} \setminus (S(\sigma_2 - \mu_2) \cup S(\mu_3)) \\
\mathcal{R}_4 &= \mathbb{C} \setminus S(\mu_3)
\end{aligned}$$

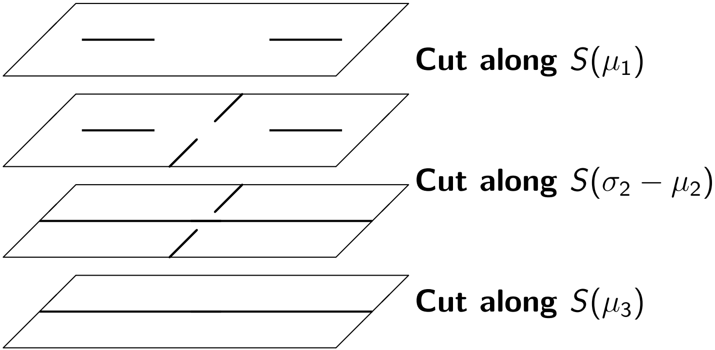
Riemann surface (Case I)

- **Case I:** $0 \in S(\mu_1), \quad 0 \notin S(\sigma_2 - \mu_2), \quad 0 \in S(\mu_3)$



Riemann surface (Case II)

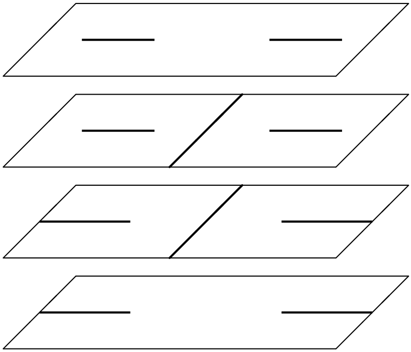
- **Case II:** $0 \notin S(\mu_1), \quad 0 \notin S(\sigma_2 - \mu_2), \quad 0 \in S(\mu_3)$



- **Cases I and II, i.e.** $S(\mu_3) = \mathbb{R}$**, are the only cases that can happen if** $\alpha \geq 0$**.**

Riemann surface (Case III)

- **Case III:** $0 \notin S(\mu_1), \quad 0 \in S(\sigma_2 - \mu_2), \quad 0 \notin S(\mu_3)$



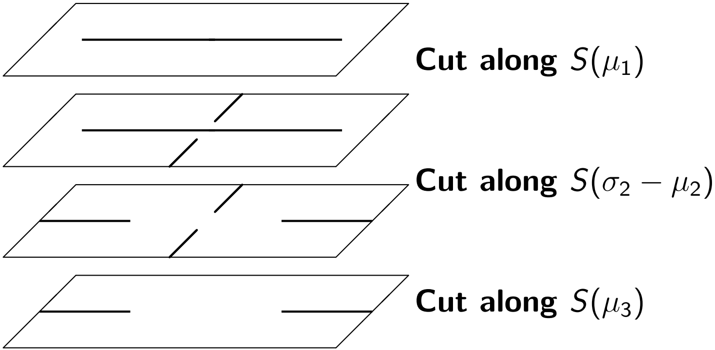
Cut along $S(\mu_1)$

Cut along $S(\sigma_2 - \mu_2)$

Cut along $S(\mu_3)$

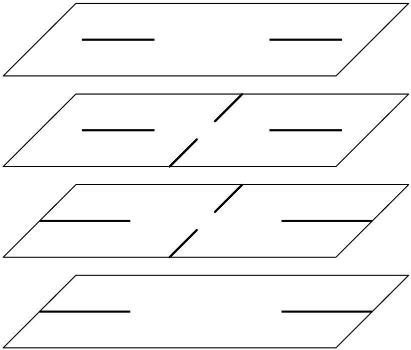
Riemann surface (Case IV)

- **Case IV:** $0 \in S(\mu_1), \quad 0 \notin S(\sigma_2 - \mu_2), \quad 0 \notin S(\mu_3)$



Riemann surface (Case V)

- **Case V:** $0 \notin S(\mu_1)$, $0 \notin S(\sigma_2 - \mu_2)$, $0 \notin S(\mu_3)$



Cut along $S(\mu_1)$

Cut along $S(\sigma_2 - \mu_2)$

Cut along $S(\mu_3)$

Meromorphic function

Proposition

The function

$$\xi_1(z) = V'(z) - \int \frac{1}{z-x}\, d\mu_1(x), \qquad z \in \mathcal{R}_1,$$

extends to a meromorphic function on the Riemann surface

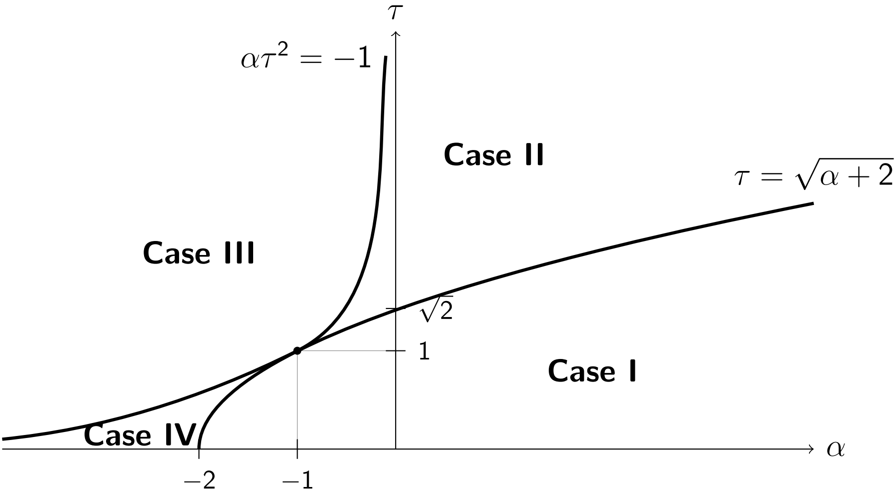
- *pole of order* $\deg V$ *at infinity on first sheet*
- *simple pole at the other point at infinity*

Consequence: ξ_1 **is solution of a quartic equation (spectral curve), which in case** $V(x) = \frac{1}{2}x^2$ **is:**

$$\xi^4 - z\xi^3 + (1+\alpha\tau^2)\xi^2 - (\alpha\tau^2 + \tau^4)z\xi + \tau^4 z^2 + C = 0$$

Phase diagram for $V(x) = \frac{1}{2}x^2$

- **All transitions between cases can be calculated for** $V(x) = \frac{1}{2}x^2$ **Duits, Geudens, K (preprint)**



Main result

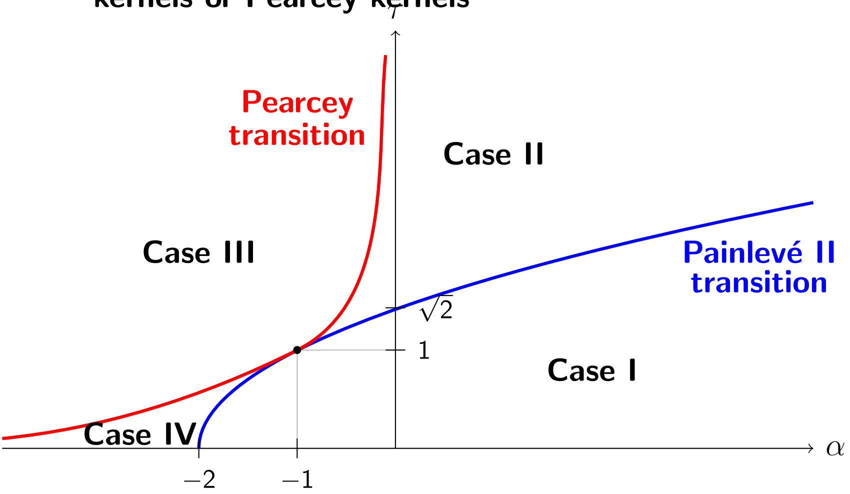
Theorem (Duits, K, Mo)

The first component μ_1 of the minimizer is equal to the limiting mean eigenvalue density of the matrix M_1 in the two-matrix model

- **We also have the usual local scaling limits from random matrix theory: sine kernel in the bulk and Airy kernel at typical edge points.**
- **We see new critical phenomena that are not possible in the one-matrix model: Pearcey kernels and more...**

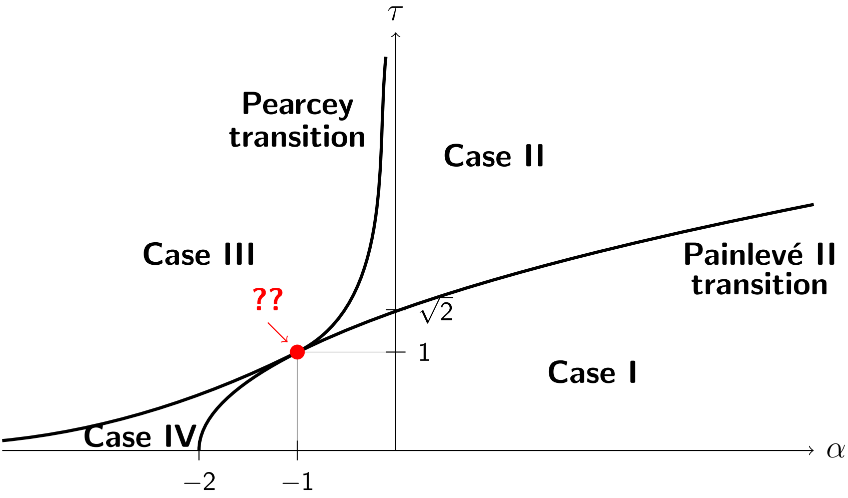
Phase diagram for $V(x) = \frac{1}{2}x^2$

- **Singular behavior at 0 corresponds to change in cases I-V, that are typically described by Painlevé II kernels or Pearcey kernels**



Special point for $\alpha = -1$, $\tau = 1$

- **One very special point in phase diagram**



Special point for $\alpha = -1$, $\tau = 1$

Theorem (Duits, Geudens (in preparation))

Local eigenvalue correlations around 0 for the special values $\alpha = -1$, $\tau = 1$ are given by the same correlation kernels as for the non-intersecting Brownian motions at a tacnode