Bivariate orthogonal polynomials and factorization

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Introduction

The representation of positive polynomials as a sum of squares of polynomials or rational functions is an important problem in mathematics and led Hilbert to pose his 17th problem. Hilbert showed that there were positive polynomials in two real variables that could not be written as a finite sum of squares of polynomials. His 17th problem was, is it possible to write every positive real multivariate polynomial as a finite sum of squares of rational functions. Emil Artin proved that you could, however the degrees of the rational functions might be very large.

Introduction

What about positive trigonometric polynomials? Here it is more convenient to use square magnitudes than squares. In this case it is known that a strictly positive multivariable trigonometric polynomial can be written as a finite sum of square magnitudes of algebraic polynomials. In other words if $Q(\bar{\theta}) \geq r > 0$ is a strictly positive trigonometric polynomial of the multivariable $\bar{\theta} = (\theta_1, \ldots, \theta_n)$ then

$$Q(\bar{\theta}) = \sum_{i=1}^{m} |p_i(\bar{z})|^2,$$

where $\bar{z} = (z_1, \ldots, z_n), \ z_j = e^{i\theta_j}, \ j = 1, \ldots, n$. The problem with the above result again is that the bound on the degrees of the polynomials p_j and the number of terms in the sum increases without bound as r decreases

Fejer-Riesz Lemma

One of the simplest factorization results is when $m = 1$ in the above sum. This result was proven in 1915 by Fejer and Riesz.

Theorem 2.1

Suppose $Q_n(\theta)$ is a positive trigonometric polynomial of degree n (i.e has terms in $\sin(k\theta)$ and $\cos(k\theta)$ for $0 \leq k \leq n$). Then

$$Q_n(\theta) = |p_n(z)|^2, \ z = e^{i\theta},$$

where $p_n(z)$ is a polynomial in z of at most degree n.

Fejer-Riesz Lemma

Proof. Since $Q_n(\theta)$ is real

$$Q_n(\theta) = \sum_{j=-n}^{n} q_j e^{ij\theta} \text{ with } q_{-j} = \bar{q}_j.$$

Set $\hat{Q}_{2n}(z) = z^n Q_n(\theta)$ $z = e^{i\theta}$. Then the constraint on the coefficients of Q_n shows that $z^{2n}\overline{\hat{Q}_{2n}(1/z)} = \hat{Q}_{2n}(z)$. Thus if z_0 is a zero of $\hat{Q}_{2n}$ then so is $1/\bar{z}_0$. This shows that

$$\hat{Q}_{2n}(z) = c\prod_j (z - z_j)(z - 1/\bar{z}_j)\prod_i (z - z_i)^2, |z_i| = 1, |z_j| \neq 1.$$

Here we have used the fact that the positivity of Q_n implies that all the zeros of $\hat{Q}_{2n}$ that are of magnitude one are of even multiplicity.

Fejer-Riesz Lemma

Since for $|z| = 1$, $Q_n(\theta) = |\hat{Q}_{2n}(z)|$ and $|z - 1/\bar{z}_j| = |1/z_j||\bar{z} - \bar{z}_j|$ we see we can choose

$$p(z) = k \prod_j (z - z_j) \prod_i (z - z_i).$$

If $Q_n > 0$ then we can choose p so that $p(z) \neq 0$ for $|z| \leq 1$. Such

polynomials are called stable. Note that the above proof uses the Fundamental Theorem of Algebra and so is difficult to generalize to higher variables.

Orthogonal polynomials

The theory of orthgonal polynomials gives an alternative proof that can be generalized.
Let μ be a positive Borel measure supported on the unit circle with an infinite number of points of increase. Let $\{\phi_i(z)\}, i = 0, 1, \ldots,$ be the unique sequence of polynomials such that

$$\phi_i(z) = k_{i,i}z^i + k_{i,i-1}z^{i-1} + \cdots k_{i,0},\ \ k_{i,i} > 0$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_i(e^{i\theta})\overline{\phi_j(e^{i\theta})}d\mu(\theta) = \delta_{i,j}.$$

Set

$$\overleftarrow{\phi}_n(z) = z^n\bar{\phi}_n(1/z).$$

$\overleftarrow{\phi}_n(z)$ is called the reverse or reciprocal polynomial.

The polynomials ϕ_i above satisfy the following difference equations.

$$\phi_n(z) = a(n)(z\phi_{n-1}(z) - \alpha_n \overleftarrow{\phi}_{n-1}(z)),$$

and

$$\overleftarrow{\phi}_n(z) = a(n)(\overleftarrow{\phi}_{n-1}(z) - z\bar{\alpha}_n \overleftarrow{\phi}_{n-1}(z)),$$

$$a(n) = \frac{k_n}{k_{n-1}}.$$

The coefficients $\{\alpha_n\}$ are called recurrence coefficients and have the property that

$$|\alpha_n| < 1.$$

Also

$$a(n)^2(1 - |\alpha_n|^2) = 1. \tag{1}$$

Recurrence formula, continued

Properties

1) $\overleftarrow{\phi}_n$ is stable (no zeros inside and on the unit circle)

2) Spectral matching, $c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ij\theta}}{|\overleftarrow{\phi}_n|^2} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta} d\mu$, $|j| \leq n$

These properties follow in a straight forward manner from the recurrence formula via for instance the Christoffel-Darboux formula,

$$\overleftarrow{\phi}_n(z)\overline{\overleftarrow{\phi}_n(z_1)} - \phi_n(z)\overline{\phi_n(z_1)} = (1 - z\bar{z}_1)\sum_{i=0}^{n-1} \phi_i(z)\overline{\phi_n(z_1)}.$$

Max Entropy

Another useful result is the Maximal Entropy theorem.
For μ a Borel measure on $\mathbb{T}$ set,

$$S = \{\mu : log \frac{du_{ac}}{d\theta} \in L^1 \ c_i \text{ fixed } |i| \leq n\}$$

and $E_\mu = \int_{\mathbb{T}} ln \frac{du_{ac}}{d\theta} d\theta$. Then E_μ has a unique maximum on S which is $d\mu = \frac{d\theta}{Q_n(\theta)}$ where Q_n is a trigonometric polynomial of degree n.
For the one variable case the theorem follows from equation 1. In general it follows from convexity.

Fejer-Riesz

The above properties of orthogonal polynomials and the Maximum Entropy theorem allows for a proof of the Fejer-Reisz theorem that does not use the Fundamental Theorem of Algebra. To see this suppose Q_n is strictly positive and let μ be absolutely continuous with respect to Lebesgue measure with density $\mu' = \frac{1}{2\pi Q_n(\theta)}$. If ϕ_n is the nth degree orthogonal polynomial associated with the above measure then $d\mu_n = \frac{1}{2\pi} \frac{d\theta}{|\overleftarrow{\phi}_n(z)|^2}$ has the same first n Fourier coefficients as μ and is the recriprocal of a trigonometric polynomial. Therefore the Maximum Entropy theorem says that $Q_n(\theta) = |\overleftarrow{\phi}_n(z)|^2$ which gives the factorization.

Fejer-Riesz

What about two varaibles?

If $Q_{n,m}(\theta, \phi)$ is a positive trigonometric polynomial in two variables of degree (n, m) (ie $Q_{n,m} = \sum_{i=-n,j=-m}^{n,m} q_{i,j} z^i w^j, \ z = e^{i\theta}, w = e^{i\phi}$) can it be written as $Q_{n,m}(\theta, \phi) = |p_{n,m}(z, w)|^2$? It is easy to see that generically this is not possible since $Q_{n,m}$ contains nm more coefficients that $p_{n,m}$ so $Q_{n,m}$ must satisfy some extra conditions. To find these conditions we turn to the theory of bivariate polynomials orthogonal on the bicircle.

Bivariate orthogonal polynomials

Given a positive measure $\mu(\theta, \phi)$ on the bicircle with moments

$$c_{l,k} = \int_{T^2} e^{-il\theta} e^{-ik\phi} du(\theta, \phi).$$

We order the monomials as $1, w, \cdots w^m, z, zw, \cdots, z^n w^m$ or $1, z, \cdots z^n, w, wz, \cdots, w^m z^n$. The first ordering is the lexicographical ordering (lex) and the second is the reverse lexicographical ordering or relex.

Bivariate orthogonal polynomials

In the lex ordering the moment matrix $C_{n,m}$ has the form

$$C_{n,m} = \begin{bmatrix} C_0 & C_{-1} & \cdots & C_{-n} \\ C_1 & C_0 & \cdots & C_{-n+1} \\ \vdots & & \ddots & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \end{bmatrix}, \tag{2}$$

where each C_i is a $(m+1)\times(m+1)$ matrix of the form

$$C_i = \begin{bmatrix} c_{i,0} & c_{i,-1} & \cdots & c_{i,-m} \\ \vdots & & \ddots & \vdots \\ c_{i,m} & & \cdots & c_{i,0} \end{bmatrix}, \qquad i = -n, \dots, n. \tag{3}$$

Bivariate orthogonal polynomials

Assume that $det(C_{n,m}) > 0$ and construct bivariate orthonormal polynomials $\phi^l_{n,m}(z,w)$, $0 \le n$, $0 \le m$, $0 \le l \le m$, satisfying,

$$\phi^l_{n,m}(z,w) = k^{n,l}_{n,m,l} z^n w^l + \sum_{(i,j) <_{\mathrm{lex}} (n,l)} k^{i,j}_{n,m,l} z^i w^j . \; k^{n,l}_{n,m,l} > 0 \quad (4)$$

$$\int_{T^2} \overline{\phi^l_{n,m}(z,w)} z^i w^j d\mu(\theta,\phi) = 0,$$

for $0 \le i < n, 0 \le j \le m$ or $i = n, 0 \le j < l$

The above equations uniquely specify $\phi^l_{n,m}$.
Polynomials orthonormal with respect to $d\mu$ but using the reverse lexicographical ordering will be denoted by $\tilde{\phi}^l_{n,m}$.
Set,

$$\Phi_{n,m} = \begin{bmatrix} \phi^m_{n,m} \\ \phi^{m-1}_{n,m} \\ \vdots \\ \phi^0_{n,m} \end{bmatrix}$$

and

$$\tilde{\Phi}_{n,m} = \begin{bmatrix} \tilde{\phi}^n_{n,m} \\ \phi^{m-1}_{n,m} \\ \vdots \\ \tilde{\phi}^0_{n,m} \end{bmatrix}$$

Recurrence Formulas

For continuous vector functions f and g we define

$$< f, g >= \int_{\mathbb{T}^2} f(\theta, \phi) g(\theta, \phi)^\dagger d\mu(\theta, \phi)$$

With this inner product we find

Recurrence Formulas

Theorem 5.1

Given $\{\Phi_{n,m}\}$ *and* $\{\tilde{\Phi}_{n,m}\}$ *the following recurrence formulas hold*

$$A_{n,m}\Phi_{n,m} = z\Phi_{n-1,m} - \hat{E}_{n,m}\overleftarrow{\Phi}^{\top}_{n-1,m}$$

$$\Phi_{n,m} + A^{\dagger}_{n,m}\hat{E}_{n,m}(A^{\top}_{n,m})^{-1}\overleftarrow{\Phi}_{n,m} = A^{\dagger}_{n,m}z\Phi_{n-1,m}$$

$$\Gamma_{n,m}\Phi_{n,m} = \Phi_{n,m-1} - \mathcal{K}_{n,m}\tilde{\Phi}_{n-1,m},$$

$$\Gamma^{1}_{n,m}\Phi_{n,m} = w\Phi_{n,m-1} - \mathcal{K}^{1}_{n,m}\overleftarrow{\tilde{\Phi}}^{\top}_{n-1,m},$$

$$\Phi_{n,m} = I_{n,m}\tilde{\Phi}_{n,m} + \Gamma^{\dagger}_{n,m}\Phi_{n,m-1},$$

$$\overleftarrow{\Phi}^{\top}_{n,m} = I^{1}_{n,m}\tilde{\Phi}_{n,m} + (\Gamma^{1}_{n,m})^{\top}\overleftarrow{\Phi}^{\top}_{n,m-1}$$

Matrix Orthogonal Polynomials

Equations with the roles of Φ and $\tilde{\Phi}$ interchanged will be denoted as tilde. The first two of the above recurrence formulas from the connection with matrix orthogonal polynomials. To see this write

$$\Phi_{i,m}(z,w) = \Phi_i^m(z) \begin{bmatrix} w^m \\ w^{m-1} \\ \vdots \\ 1 \end{bmatrix}$$

for $i = 0, 1, \ldots n$ where Φ_i^m are $m+1 \times m+1$ matrix polynomials of degree i in z satisfying

$$\int_{-\pi}^{\pi} \Phi_j^m(z) dW(\theta) \Phi_k^m(z)^\dagger = I_{m+1}\delta_{j,k} \; z = e^{i\theta}.$$

Matrix Orthogonal Polynomials

Here

$$dW(\theta) = \int_{-\pi}^{\pi} \begin{bmatrix} w^m \\ w^{m-1} \\ \vdots \\ 1 \end{bmatrix} d\mu(\theta, \phi) \begin{bmatrix} w^m \\ w^{m-1} \\ \vdots \\ 1 \end{bmatrix}^{\dagger} \quad w = e^{i\phi}.$$

Note that dW is an $m+1 \times m+1$ Toeplitz matrix.

Two variable factorization

The above structures allow us to prove the following two variable Fejer-Reisz factorization result.

Theorem 5.2

Suppose that $Q_{n,m}$ is a strictly positive bivariate trigonometric polynomial of degree (n, m). Then $Q_{n,m}(\theta, \phi) = |p(z, w)|^2$ where $p(z, w)$, $z = e^{i\theta}$, $w = e^{i\varphi}$ is a polynomial of degree at most (n, m) and $p(z, w) \neq 0$ for $|z| = 1$ and $|w| \leq 1$ if and only if the coefficients associated with $\frac{d\varphi d\theta}{4\pi^2 Q(\theta,\varphi)}$ satisfy $\mathcal{K}_{n,m}[\tilde{\Gamma}^1_{n,m}\tilde{\Gamma}^\dagger_{n,m}]^j(\mathcal{K}^1_{n,m})^T = 0$ for $j = 0, 1, \ldots, n-1$. In this case

$$p(z, w)\overline{p(1/\bar{z}, 0)} = \Phi_{n,m}(z, w)\overline{\Phi_{n,m}(1/\bar{z}, 0)}.$$

Also,

Factorization Fejer-Riesz II

Theorem 5.3

Suppose that $Q_{n,m}$ is a strictly positive bivariate trigonometric polynomial of degree (n, m). Then $Q_{n,m}(\theta, \phi) = |p_{n_1,m_1}(z, w)q_{n-n_1,m-m_1}(1/z, w)|^2$ where $p_{n_1,m_1}(z, w)$ is a stable polynomial of degree (n_1, m_1) and $q_{n-n_1,m-m_1}(z, w)$ is a stable polynomial of degree $(n - n_1, m - m_1)$ if and only if $K_{n,m}[\tilde{\Gamma}^1_{n,m}\tilde{\Gamma}^\dagger_{n,m}]^j(\mathcal{K}^1_{n,m})^\top = 0$ for $j = 0, 1, \ldots, n - 1$ and $K^\dagger_{n,m}[\Gamma^1_{n,m}\Gamma^\dagger_{n,m}]^i(\mathcal{K}^1_{n,m})^\top = 0$ for $i = 0, 1, \ldots, m - 1$

Factorization

For $j = 0$ we see that $K_{n,m}(\mathcal{K}^1_{n,m})^\top = 0$ so that $AB = BA$ where $A = K^\dagger_{n,m} K_{n,m}$ and $B = (\mathcal{K}^1_{n,m})^\top \bar{\mathcal{K}}^1_{n,m}$. This means roughly speaking that the singular value decompositions of $\mathcal{K}_{n,m}$ and $(\mathcal{K}^1_{n,m})^\top$ have common eigenfunctions and the diagonal parts are orthogonal. This observation plays an important role in the proofs of the above Theorems.

Factorization

We now have the following result for two dimensional spectral theory,

Theorem 5.4

Suppose $\hat{E}_{i,j} = 0$ *for* $j \geq m, \ i \geq n+1$, *and* $\hat{\tilde{E}}_{i,j} = 0$ *for* $i \geq n, \ j \geq m+1$. *Then it is necessary and sufficient that* $d\mu = 1/|p_{n,m}(z,w)|^2 d\theta d\phi$ *where* $p_{n,m}(z,w)$ *is a polynomial of degee* $n_p \leq n$ *in* z *and* $m_p \leq m$ *in* w *with*

$$p_{n,m}(z,w) = p_{n_1,m_1}(z,w) q_{n_2,m_2}(1/z,w).$$

Here $p_{n_1,m_1(z,w)}$ *and* $q_{n_2,m_2}(z,w)$ *are stable polynomials of degree* (n_1,m_1), (n_2,m_2) *in* (z,w) *respectively with* $n_1+n_2 = n_p$ *and* $m_1+m_2 = m_p$.

Factorization

Structure of orthogonal polynomials.

Theorem 5.5

Let $\phi^i_{n_1,m_1}$ $i = 0, \cdots m_1$ be the orthonormal polynomials associated with p_{n_1,m_1} and $\phi^i_{n_2,m_2}$ $i = 0, \cdots m_2 - 1$ be the orthonormal polynomials associated with q_{n_2,m_2}. Then under the conditions of Theorem (5.4) the polynomials $z^{n_2}q(1/z, w)\phi^i_{n_1,m_1}(z, w)$ and $w^{m_2}\phi^j_{n_2,m_2}(z, 1/w)z^{n_1}w^{m_1}p(1/z, 1/w)$ form an orthonormal basis for the space of polynomials of degrees at most (n,m), perpendicular to $\{z^i w^j : 0 \leq i < n, 0 \leq j \leq m\}$ w.r.t $d\mu$. Hence, $\Phi_{n,m}$ can be related to this basis by a unitary transformation.

Factorization

Special cases.

Theorem 5.6

Suppose $\mathcal{K}_{n,m} = 0$ then $p_{n,m}$ is stable (ie $p_{n,m}(z,w) \neq 0,\ |z| \leq 1,\ |w| \leq 1$). If $\mathcal{K}^1_{n,m} = 0$ then $p_{n,m}(1/z,w)$ is stable.

In this case spectral factorization has a geometric component. For instance from the definition of $\mathcal{K}$ we see that $\mathcal{K}_{n,m} = 0$ if and only if $< \Phi_{n,m-1}, \tilde{\Phi}_{n-1,m} >= 0$