Rearrangement of series. The theorem of Levy-Steiniz

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- Rearrangement of series.
- Surprising phenomenon in mathematical analysis: Infinite sums of numbers do NOT satisfy the commutative property.

- **Series** $\sum a_k$.
  Sequence of real number $a_1, a_2, ..., a_k, ...,$ called terms of the series. We want to associate to them a **sum**.
- **Idea of Cauchy**.
- **Partial Sums**
  $s_1 := a_1$, $s_2 := a_1 + a_2$,..., $s_k := a_1 + a_2 + ... + a_k$.
- **The series converges** if the limit $\text{lím}\, s_k = s$ exists and this limit is called the **sum** of the series $s = \sum_{k=1}^{\infty} a_k$.
- $\sum x^k = 1 + x + x^2 + ...$ converges if and only if $|x| < 1$.
  Its sum is $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

Aspects to study about series.

(1) **Convergence.** $\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$ (**Euler**).

(2) **Asymptotic behaviour.**
The harmonic series $\sum \frac{1}{k}$ diverges (**Euler**). Its partial sums s_k behave asymptotically like $\log k$. This means

$$\lim_{k\to\infty} \frac{s_k}{\log k} = 1.$$

Euler proved also that the series $\sum \frac{1}{p}$, the sum extended to the prime numbers p diverges.

(3) An aspect that is exclusive to series is **rearrangement**.

Rearrangement of series

- A **rearrangement** of the series $\sum a_k$ is the series $\sum a_{\pi(k)}$, where
$$\pi : \mathbb{N} \to \mathbb{N}$$
is a bijection.
- A series $\sum a_k$ is **unconditionally convergent** if the series $\sum a_{\pi(k)}$ converges for each bijection π.
- If the series $\sum a_k$ converges, the **set of sums** is
$$S(\sum a_k) := \{x \in \mathbb{R} \mid x = \sum_{k=1}^{\infty} a_{\pi(k)} \ \text{ for some } \ \pi\}$$
It is the set of sums of all the rearrangements of the series.

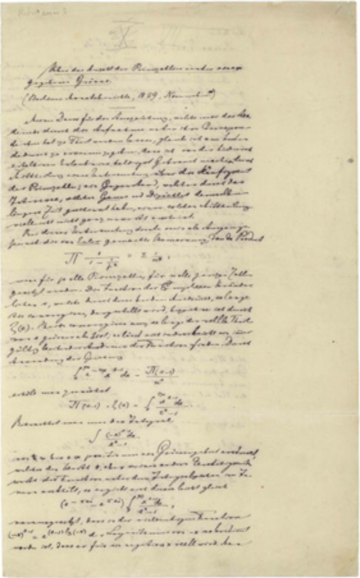
The Theorem of Riemann

Theorem (Riemann, 1857)

Let $\sum a_k$ be a series of real numbers.

- $\sum a_k$ is unconditionally convergent if and only if $\sum |a_k|$ is convergent, that is the series is **absolutely convergent**.
- If $\sum a_k$ converges, but not unconditionally, then

$$S(\sum a_k) = \mathbb{R}.$$



Riemann (1826-1866). Riemann integral, Riemann's surfaces, the Cauchy Riemann equation, the theorem of Riemann Lebesgue, the theorem of Riemann's function in complex analysis, Riemann's zeta function,...

The center of Mathematics between 1800 y 1933 was **Göttingen (Germany)**.

Gauss, Dirichlet, Riemann, Hilbert y Klein were there.

Idea of the proof:
If $\sum a_k$ converges absolutely, Cauchy's criterion ensures that every rearrangement converges to the same sum.

Suppose that $\sum a_k$ converges but not absolutely. Let p_k y q_k be the positive and negative terms of the series respectively. Possible cases:

- $\sum p_k$ converges, $\sum q_k$ converges, then $\sum |a_k|$ converges.
- $\sum p_k = \infty$, $\sum q_k$ converges, then $\sum a_k = \infty$.
- $\sum p_k$ converges, $\sum q_k = -\infty$, then $\sum a_k = -\infty$.

Thus $\sum p_k = \infty$ y $\sum q_k = -\infty$.

Fix $\alpha \in \mathbb{R}$, select the first $n(1)$ with $p_1 + ... + p_{n(1)} > \alpha$; then the first $n(2)$ with $p_1 + ... + p_{n(1)} + q_1 + ... + q_{n(2)} < \alpha$.
Since $\lim a_k = 0$, the rest of the proof is ε-δ.

The alternate harmonic series.

The alternate harmonic series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \log 2.$

Leibniz's criterion shows that this series is convergent. It is not absolutely convergent.

The result was known to **Mercator (S. XVII)**.

There are many different proofs, for example by a **theorem of Abel** on power series using the development of $\log(1 + x)$.

The alternate harmonic series.

An elementary proof: Put $I_n := \int_0^{\pi/4} \mathrm{tg}^n\, x dx$. We have:

(1) $(I_n)_n$ is decreasing.

(2) $I_n = \frac{1}{n-1} - I_{n-2}$. Integrating by parts.

(3) $\frac{1}{2(n+1)} \leq I_n \leq \frac{1}{2(n-1)}$.

(4) Using induction in (2) and $I_1 = \frac{1}{2}\log 2$, we get

$$\frac{1}{4(n+1)} \leq |I_{2n+1}| = \Big|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k} - \frac{1}{2}\log 2\Big| \leq \frac{1}{4n}.$$

Multiplying by 2 we obtain the result.

The alternate harmonic series.

The rearrangement of Laurent.

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} ... =$$

$$= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} ... =$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} ... =$$

$$= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...) = \frac{1}{2}\log 2.$$

The alternate harmonic series.

A rearrangement of the alternate harmonic series is called **simple** if the positive and negative terms separately are in the same order as in the original series. For example, Laurent's rearrangement is simple.

In a simple rearrangement we denote by r_n the number of positive terms between the first n of the rearrangement.

Theorem of Pringsheim, 1883

A simple rearrangement $\sum a_{\pi(k)}$ of the alternate harmonic series converges if and only if $\text{lím}_{n\to\infty} \frac{r_n}{n} =: \alpha < \infty$.
In this case $\sum_{k=1}^{\infty} a_{\pi(k)} = \log 2 + \frac{1}{2}\log(\alpha(1-\alpha)^{-1})$.

For the Laurent's rearrangement we have $\alpha = 1/3$ y

$$\log 2 + \frac{1}{2}\log(\frac{1}{3}\frac{3}{2}) = \frac{1}{2}\log 2.$$

Series of vectors.

What happens if we consider series of vectors?

- Example in $\mathbb{R}^2$: $\sum((-1)^{k+1}\frac{1}{k}, 0)$.
- The set of sums is $\mathbb{R} \times \{0\}$. It is not all the space $\mathbb{R}^2$, but it is an affine subspace $\mathbb{R}^2$.
- This phenomenon was observed by Levy for $n = 2$ in 1905 and by Steinitz for $n \geq 3$ in 1913.

The Theorem Levy Steinitz. Notation.

- E is a **real** locally convex Hausdorff space.
- **Examples:** $\mathbb{R}^n$, $\ell_p, 1 \leq p \leq \infty$, $L_p, 1 \leq p \leq \infty$ (Banach spaces), $H(\Omega)$, $C^\infty(\Omega)$ (Fréchet spaces: metrizable and complete), $\mathcal{D}$, $\mathcal{D}'$, $H(K)$, $\mathcal{A}(\Omega)$,...(more complicated spaces).
- $\sum u_k$ is a convergent series and $S(\sum u_k)$ is its **set of sums** (of all its convergent rearrangements).
- **Set of summing functionals**

$$\Gamma(\sum u_k) := \{x' \in E' \mid \sum_1^\infty |\langle x', u_k\rangle| < \infty\} \subset E'.$$

- The **annihilator** of $G \subset E'$ is $G^\perp := \{x \in E \mid \langle x, g\rangle = 0 \ \forall g \in G\}$.

The Theorem Levy Steinitz.

The Theorem Levy Steinitz. 1905, 1913.

If $\sum u_k$ is a convergent series of vectors in $\mathbb{R}^n$, then

$$S(\sum u_k) = \sum_{1}^{\infty} u_k + \Gamma(\sum u_k)^{\perp}$$

is an affine subspace of $\mathbb{R}^n$.

P. Rosenthal, in an article in the American Mathematical Monthly in 1987 explaining this theorem, remarked that it is a beautiful result, which deserves to be better known, but that the difficulty of its proof is out of proportion of the statement.

The inclusion "$\subset$" in the statement is easy and holds in general:

Let $x = \sum_1^\infty u_{\pi(k)} \in S(\sum u_k)$.

We want to see that $x - \sum_1^\infty u_k \in \Gamma(\sum u_k)^\perp$.

To do this, fix $x' \in \Gamma(\sum u_k)$.

By Riemann's theorem, we get

$$\langle x', x - \sum_1^\infty u_k \rangle = \sum_1^\infty \langle x', u_{\pi(k)} \rangle - \sum_1^\infty \langle x', u_k \rangle = 0,$$

since the series $\sum \langle x', u_k \rangle$ is absolutely convergent.

Idea of the proof of the other inclusion: Let E be a complete metrizable space

(A)

$S(\sum u_k) \subset S_e(\sum u_k)$.

Expanded set of sums

$$S_e(\sum u_k) := \{x \in E \mid \exists \pi \ \exists (j_m)_m \ : \ x = \lim_{m \to \infty} \sum_1^{j_m} u_{\pi(k)}\}.$$

(B)

$S_e(\sum u_k) = \sum_1^\infty u_k + \cap_{m=1}^\infty \overline{Z_m}$.

$$Z_m = Z_m(\sum u_k) := \{\sum_{k \in J} u_k \mid J \subset \{m, m+1, m+2, ...\} \text{ finite}\}.$$

Idea of the proof of the other inclusion: Continued:

(C)

$$\sum_{1}^{\infty} u_k + \cap_{m=1}^{\infty} \overline{Z_m} \subset \sum_{1}^{\infty} u_k + \cap_{m=1}^{\infty} \overline{\mathrm{co}(Z_m)}.$$

$\mathrm{co}(C)$ is the convex hull of C.

(D)

$$\sum_{1}^{\infty} u_k + \cap_{m=1}^{\infty} \overline{\mathrm{co}(Z_m)} = \sum_{1}^{\infty} u_k + \Gamma(\sum u_k)^{\perp},$$

by the Hahn-Banach theorem.

The problem is to find conditions to ensure that the inclusions **(A)** y **(C)** are equalities.

The Theorem Levy Steinitz.

The equality in **(A)** follows in the finite dimensional case from the following lemma.

Lemma of polygonal confinement of Steinitz

For every real Banach space E of finite dimension m there is a constant $0 < C(E) \leq m$ such that for every finite set of vectors $x_1, x_2, ..., x_n$ satisfying $\sum_1^n x_k = 0$ there s a bijection σ on $\{1, 2, ..., n\}$ such that

$$||\sum_{j=1}^{r} x_{\sigma(j)}|| \leq C(E) \max_{j=1,\dots n} ||x_j||$$

for all $r = 1, 2, ...n$.

The exact value of the constant $C(E)$ is unknown even for Hilbert spaces of finite dimension $m > 2$. For $m = 2$, $C(\ell_2^2) = \frac{\sqrt{5}}{2}$.

The equality in **(C)** follows in the finite dimensional case from the following lemma.

Round-off coefficients Lemma.

Let E be a real Banach space of finite dimension m.

Let $x_1, x_2, ..., x_n$ be a finite set of vectors such that $||x_j|| \leq 1$ for all $j = 1, ..., n$.

For each $x \in \mathrm{co}(\sum_{k \in I} x_k \mid I \subset \{1, ..., n\})$ there is $J \subset \{1, 2, ..., n\}$ such that $||x - \sum_{k \in J} x_k|| \leq \frac{m}{2}$.

Series in infinite dimensional Banach spaces.

The study of series in infinite dimensional spaces was initiated by **Orlicz** in 1929-1930.

Banach and his group used to meet in the Scottish Café in Lvov (now Ukraine). The problems they formulated were recorded in the **Scottish Book**, that was saved and published by Ulam.

Problem 106: Does a result analogous to the Levy Steinitz theorem hold for Banach spaces of infinite dimension? The prize was a bottle of wine; smaller by the way than the prize for the approximation problem of Mazur, that was solved by Enflo. In that case the prize was a goose.

The negative answer was obtained by Marcinkiewicz with an example in $L_2[0,1]$.

The Scottish Cafe, Lvov.

The Scottish Cafe, Lvov. 2010.

The example of Marcinkiewicz.

Consider the following functions in $L_2[0,1]$. Here χ_A is the characteristic function of A.

$$x_{i,k} := \chi_{[\frac{k}{2^i}, \frac{k+1}{2^i}]}, \quad y_{i,k} := -x_{i,k}, \ 0 \leq i < \infty, 0 \leq k < 2^i.$$

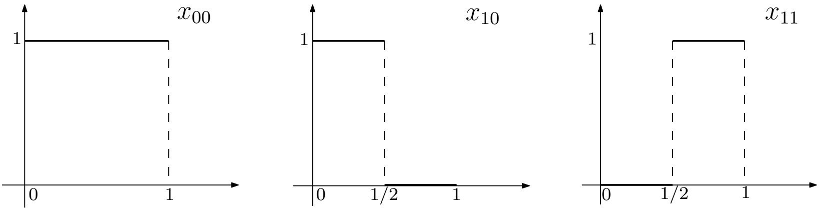
Clearly $||x_{i,k}||^2 = 2^{-i}$ for each i, k. One has

$$(x_{0,0} + y_{0,0}) + (x_{1,0} + y_{1,0}) + (x_{1,1} + y_{1,1}) + (x_{2,0} + y_{2,0}) + ... = 0$$

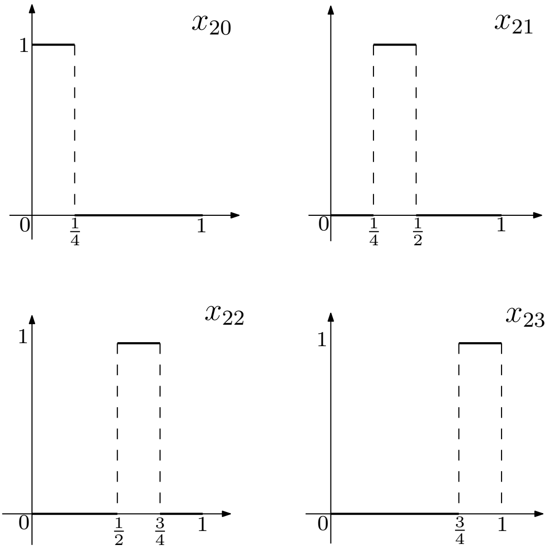
$$x_{0,0} + (x_{1,0} + x_{1,1} + y_{0,0}) + (x_{2,0} + x_{2,1} + y_{1,0}) + (x_{2,2} + x_{2,3} + y_{1,1}) + ... = 1$$

No rearrangement converges to the constant function $1/2$, since all the partial sums are functions with entire values.

The example of Marcinkiewicz.



The example of Marcinkiewicz.



Dvoretsky-Rogers Theorem. 1950.

A Banach space E is finite dimensional if and only if every unconditionally convergent series in E is absolutely convergent.

This is a very important result in the theory of nuclear locally convex spaces of **Grothendieck** and in the theory of absolutely summing operators of **Pietsch**.

Example. In ℓ_2, we set $u_k := (0,..,0,1/k,0,...)$. The series $\sum u_k$ is not absolutely convergent since $\sum_1^\infty ||u_k|| = \sum_1^\infty \frac{1}{k} = \infty$. But,

$$\sum_1^\infty u_k = (1, 1/2, 1/3, ..., 1/k, ...)$$

unconditionally in ℓ_2.

Series in infinite dimensional Banach spaces.

Theorem of Mc Arthur. 1954.

Every Banach space E of infinite dimension contains a series whose set of sums reduces to a point, but is not unconditionally convergent.

Idea in ℓ_2. Denote by e_i the canonical basis.

$$e_1 - e_1 + (1/2)e_2 - (1/2)e_2 + (1/2)e_2 - (1/2)e_2 + (1/4)e_3 - ... = 0.$$

We have 2^n terms of the form $2^{-n+1}e_n$ with alternate signs.

If a rearrangement converges, its sum must be 0, as can be seen looking at each coordinate. However, it is not unconditionally convergent, for if it were, then $(2, 2, 2, ...) \in \ell_2$.

For an arbitrary Banach space, one uses basic sequences.

Series in infinite dimensional Banach spaces.

Theorem of Kadets and Enflo. 1986-89.

Every Banach space E of infinite dimension contains a series whose set of sums consists exactly of two different points.

Theorem of J.O. Wojtaszczyk. 2005.

Every Banach space E of infinite dimension contains a series whose set of sums is an arbitrary finite set which is affinely independent.

The theorem of Levy Steinitz fails in a drastic way for infinite dimensional Banach spaces.

Series in infinite dimensional Banach spaces.

Theorem of Ostrovski. 1988.

There is a series in $L_2([0,1]\times[0,1])$ whose set of sums is not closed.

Problem.

Is there a series $\sum u_k$ in a Banach space whose set of sums $S(\sum u_k)$ is a non-closed affine subspace?

Theorem.

Every separable Banach space contains a series whose set of sums is all the space.

Problem.

Is it possible to extend the theorem of Levy Steinitz for some infinite dimensional spaces?

YES.

A locally convex Hausdorff space E is called **nuclear** if every unconditionally convergent series is absolutely convergent. For Fréchet of (DF)-spaces this coincides with the original definition of Grothendieck. In general this is not the case.

Examples. $H(\Omega)$, $C^{\infty}(\Omega)$, S, S', $H(K)$, $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{A}(\Omega)$.

The theorem of Banaszczyk.

Theorem of Banaszczyk. 1990, 1993.

Let E be a Fréchet space. The following conditions are equivalent:

(1) E is nuclear.

(2) For each convergent series $\sum u_k$ in E we have

$$S(\sum u_k) = \sum_{1}^{\infty} u_k + \Gamma(\sum u_k)^{\perp}.$$

This is a very deep result. Both directions are difficult. extensions of the lemmas of confinement and of rounding-off coefficients with Hilbert-Schmidt operators, a characterization of nuclear Fréchet spaces with volume numbers, topological groups, etc are needed.

Series in non-metrizable spaces.

Bonet and Defant studied in 2000 the set of sums of series in non-metrizable spaces and, in particular, in (DF)-spaces, like the space S' of Schwartz or the space $H(K)$ of germs of holomorphic functions on the compact set K in the complex plane.

The notation $E = \mathrm{ind}_n E_n$ means that E is the increasing union of the Banach spaces $E_n \subset E_{n+1}$ with continuous inclusions, and E is endowed with the finest locally convex topology such that all the inclusions $E_n \subset E$ are continuous.

Series in non-metrizable spaces.

Theorem of Bonet and Defant. 2000.

Let $\sum u_k$ be a convergent series in the nuclear (DF)-space $E = \mathrm{ind}_n E_n$ (then it converges in a Banach step $E_{n(0)}$). The following holds:

(a) $S(\sum u_k) = \sum_1^\infty u_k + \Gamma_{loc}^{\perp}(\sum u_k)$, where

$$\Gamma_{loc}^{\perp}(\sum u_k) :=$$

$$\bigcup_{n \geq n(0)} \{x \in E_n \mid \langle x, x' \rangle = 0 \ \forall x' \in E_n' \text{ with } \sum_1^\infty |\langle u_k, x' \rangle| < \infty\}$$

is a subspace of E.

(b) If E is not isomorphic to the direct sum φ of copies of $\mathbb{R}$, then there is a convergent series in E whose set of sums is a non-closed subspace of E.

Series in non-metrizable spaces.

Theorem of Bonet and Defant. 2000.

Let $E = \mathrm{ind}_n E_n$ be a complete (DF)-space such that every convergent sequence in E converges in one of the Banach spaces E_n. If we have

$$S(\sum u_k) = \sum_1^\infty u_k + \Gamma_{loc}^{\perp}(\sum u_k)$$

for every convergent series $\sum u_k$ in E, then the space E is nuclear.

Series in non-metrizable spaces.

- The proof of the theorem requires new improvements in the lemmas of confinement and round-off.
- The techniques of proof for the positive part can be utilized for more general spaces, including the space of distributions $\mathcal{D}'$ or the space of real analytic functions $\mathcal{A}(\Omega)$, thus obtaining that the set of sums of a convergent series is an affine subspaces that need not be closed.
- The result about nuclear (DF)-spaces not isomorphic to φ requires deep results due to Bonet, Meise, Taylor (1991) and Dubinski, Vogt (1985) about the existence of quotients of nuclear Fréchet spaces without the bounded approximation property and their duals.

Other open problems.

- Does every non-nuclear Fréchet space contain a convergent series $\sum u_k$ such that its set of sums consists exactly of two points?
- Improve the converse for non-metrizable spaces.
- Find concrete spaces E and conditions on a convergent series $\sum u_k$ in E to ensure that the set of sums $S(\sum u_k)$ has exactly the form of the Theorem of Levy and Steinitz. Chasco and Chobayan have results of this type for spaces L_p of p-integrable functions.

Some references.

W. Banaszczyk, The Steinitz theorem on rearrangement of series for nuclear spaces, J. reine angew. Math. 403 (1990), 187–200.

W. Banaszczyk, Rearrangement of series in nonnuclear spaces, Studia Math. 107 (1993), 213–222.

J. Bonet, A. Defant, The Levy-Steinitz rearrangement theorem for duals of metrizable spaces, Israel J. Math. 117 (2000), 131-156.

M.I. Kadets, V.M. Kadets, Series in Banach Spaces Operator Theory: Advances and Applications 94, Birkhäuser Verlag, Basel 1997.

P. Rosenthal, The remarkable theorem of Levy and Steinitz, Amer. Math. Monthly 94 (1987), 342-351.