Von Neumann, 1950: Let $A : H \to H$ a contraction in the Hilbert space H. Then there exists an embedding $H \hookrightarrow H^a$ of H in some enlarged Hilber space H^a and an unitary operator $U : H^a \mapsto H^a$ such that

$$A^n u = \Pi U^n u, \qquad u \in H, \quad n \geq 0,$$

where $\Pi : H^a \mapsto H$ stands for the ortogonal poryector on H.

In consequence, for any $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in \mathcal{A}(\mathbb{D})$ there hold

$$f(A) = \sum_{n=0}^{+\infty} a_n A^n = \Pi \left(\sum_{n=0}^{+\infty} a_n U^n \right) = \Pi f(U) \text{ and } \|f(A)\| \leq \|f\|_{\mathbb{D}}.$$

Nagy-Foias, 1965: Let $A : H \to H$ an operator such that $W(A) \subset \mathbb{D}$. Then there exists an embedding $H \hookrightarrow H^a$ of H in some enlarged Hilber space H^a and an unitary operator $U : H^a \mapsto H^a$ such that

$$A^n u = 2\Pi U^n u, \qquad u \in H, \quad n \geq 1,$$

where $\Pi : H^a \mapsto H$ stands for the ortogonal poryector on H.

In consequence, for any $f(z) = \sum_{n=1}^{+\infty} a_n z^n \in \mathcal{A}(\mathbb{D})$ (i.e., $f(0) = 0$) there hold

$$f(A) = \sum_{n=1}^{+\infty} a_n A^n = 2\Pi \left(\sum_{n=1}^{+\infty} a_n U^n \right) = 2\Pi f(U) \text{ and } \|f(A)\| \leq 2\|f\|_{\mathbb{D}}.$$

Actually, $W(A) \subset \mathbb{D}$ is equilvalento to saying that A admits a 2-unitary dilation. Okubo, 1975, showed thats this is equivalent to the condition

$$A = 2P \sin(B) P^{-1}$$

where $\|P\| \cdot \|P^{-1}\| \leq 1$ and B is sefl-adjoint.

By using this result, Badea, 2005, proved that

under the hypothesis $W(A) \subset \mathbb{D}$ we have

$$\|f(A)\| \leq 2\|f\|_{\mathbb{D}}, \qquad f \in \mathcal{A}(\mathbb{D}),$$

thus removing the restriction $f(0) = 0$. This is precisely *Crouzeix's conjecture* for the the disk.

F. and B. Delyon, 2000, showed that gvien an open and convex set $\Omega \subset \mathbb{C}$, there exists $C_\Omega > 0$ such that

$$\|f(A)\| \leq C_\Omega \|f\|_\Omega, \qquad f \in \mathcal{A}(\Omega),$$

whenever $A : HtoH$ is a linear operator on a Hilber space H with $W(A) \subset \Omega$.

A great remark is that they realized that if $W(A)$ is surrounded by a smooth path Γ then

$$\nu(z)(zI - A)^{-1} + \overline{\nu(z)}(\bar{z}I - A^*)^{-1}, \quad z \in \Gamma$$

is a positive operator (self-adjoint + quadratic for positive definite). Here $\nu(z)$, $z \in \Gamma$, stands for the outwards unit vector at z (w.r.t. Γ, of course).

For any mapping $\varphi : \Gamma \to \mathbb{C}$ surroinding $W(A)$ we define

$$C(\varphi, A) = C_\Gamma(\varphi, A) = \frac{1}{2\pi i} \int_\Gamma \varphi(z)(zI - A)^{-1}\, dz,$$

which is a bounded and linear operator on H. We are not entitle to write

$$\varphi(A) = C(\varphi, A)$$

unless φ admits a holomorpchic extension, continuous up to the boundary, to the interior region of Γ.

The double layer potencial of $\varphi : \Gamma \to \mathbb{C}$ is the mapping $M\varphi$: $\mathrm{int}(\Gamma) \to \mathbb{C}$ defined by

$$M\varphi(z) = \int_\Gamma \varphi(\zeta)\mu(\zeta, z)\, ds(\zeta), \qquad z \in \mathrm{int}(\Gamma)$$

whith kernel

$$\mu(\zeta, z) = \frac{1}{2\pi}\left(\frac{\nu(\zeta)}{\zeta - z} + \frac{\overline{\nu(\zeta)}}{\bar{\zeta} - z}\right), \quad z \in \mathrm{int}(\Gamma), \zeta \in \Gamma$$

There holds

$$\mu(\zeta, z) = \frac{1}{\pi}\frac{d}{ds}\arg(\varphi(\zeta) - \varphi(z)), \quad z \in \mathrm{int}(\Gamma), \zeta \in \Gamma.$$

We observe that $\mu(\zeta, z)$ is also defined for z, $\zeta \in \Gamma$ with $z \neq \zeta$.

The double layer potencial $M\varphi$ of φ is harmonic in the interior region int(Γ).

Lemma: If int(Γ) is convex then $\mu(\zeta, z) \geq 0$, for $z, \zeta \in \Gamma$ with $z \neq \zeta$, and

$$\int_\Gamma \mu(\zeta, z)\, ds(\zeta) = 2,\ z \in \text{int}(\Gamma); \quad \int_{\Gamma \setminus \{z\}} \mu(\zeta, z)\, ds(\zeta) = 1,\ z \in \Gamma.$$

Runge-Kutta time discretizations of abstract parabolic Volterra equations with bad initial data

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XXIV CEDYA, Cádiz, June 8-12, 2015

Outline of the talk

1. The abstract parabolic equation

2. Runge-Kutta convolution quadrature

3. Runge-Kutta time discretization

4. Representation of the numerical solution

5. Applications

2. The method for the convolution equation

The problem:

(1) $$u(t) = u_0 + \int_0^t A(t-s)u(s)ds, \quad t > 0,$$

where $A(t) : D \subset X \to X$, $t \geq 0$, is a locally integrable family of linear and bounded operators in a complex Banach space X and $u_0 \in X$ is a given initial data.

H1: There exists a strongly continuous family $E(t) : X \to X$, $t \geq 0$, of linear and bounded operators, with $E(0) = I$, such that $E(t)D \subset D$, $t > 0$ and, for each $u_0 \in D$, the mapping $s \to E(s)u_0$ belongs to $L^1_{loc}((0, +\infty), D)$ and satisfies

(2) $$E(t)u_0 = u_0 + \int_0^t A(t-s)E(s)u_0\, ds, \qquad t > 0.$$

Examples:

Scalar type equations [Prüss, 1993], where $A(t) = a(t)A$, $t \geq 0$, $a : (0, \infty) \to \mathbb{C}$ is locally integrable and $A : D \subset X \to X$ is a fixed operator.

$a(t) \equiv 1$, corresponds to the IVP $u'(t) = Au(t)$, $t \geq 0$, with initial condition u_0.

$a(t) = t$, corresponds to second order IVPs and cosine families,

$a(t) = t^{-\alpha}$, $0 < \alpha < 1$ corresponds to fractional wave equations.

More general kernels appear in applications such as viscoelasticity [Prüss, 1993] and when dealing with transparent boundary conditions (see Numerical illustration).

Additional hypotheses

H2: The kernel is of exponential growth, i.e.

$$\int_0^{+\infty} \mathrm{e}^{-\omega t} \|A(t)\|_{D \to X} \, \mathrm{d}t < +\infty,$$

for some $\omega \in \mathbb{R}$, so that A admits a Laplace transform $\widetilde{A}(\lambda)$, $\mathrm{Re}\, \lambda > \omega$.

H3: The operator $(I - \widetilde{A}(\lambda)) : D \to X$ admits a bounded inverse

$$(I - \widetilde{A}(\lambda))^{-1} : X \to D \subset X.$$