Operator Algebras: an extraordinary legacy of John von Neumann

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In mathematics you don't understand things. You just get used to them.

–John von Neumann, reply to a physicist at Los Alamos who had said "I don't understand the method of characteristics."

Introduction

John von Neumann developed the theory of operator algebras on a Hilbert space in the 30's and 40's of the past century, in a series of papers, a good part of them in collaboration with F. J. Murray.

A monumental series

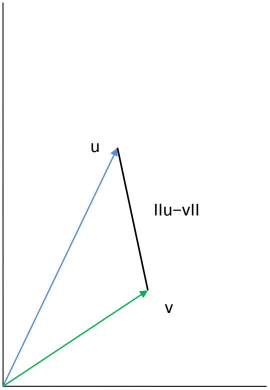
F. J. Murray, J. von Neumann, On rings of operators.
Ann. of Math. (2) 37 (1936), no. 1, 116–229.

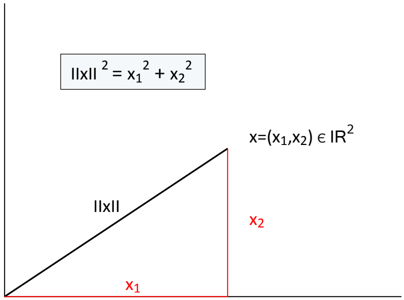
F. J. Murray, J. von Neumann, On rings of operators. II.
Trans. Amer. Math. Soc. 41 (1937), no. 2, 208–248.

J. von Neumann, On rings of operators. III.
Ann. of Math. (2) 41 (1940). 94–161.

F. J. Murray, J. von Neumann, On rings of operators. IV.
Ann. of Math. (2) 44 (1943). 716–808.

Some definitions





Hilbert space

Hilbert space of dimension n: $\mathbb{C}^n$ with inner product $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$, which induces the norm:

$$\|(x_1, \dots, x_n)\|^2 = |x_1|^2 + \cdots + |x_n|^2.$$

Infinite dimensional Hilbert space:
$\ell^2 = \{(x_1, x_2 \dots,) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, with inner product

$$\langle (x_i), (y_i) \rangle = \sum_{i=1}^{\infty} x_i \overline{y}_i,$$

which induces the norm:

$$\|(x_i)\|^2 = \sum_{i=1}^{\infty} |x_i|^2.$$

The space of all linear operators $\mathbb{C}^n \to \mathbb{C}^n$ is the matrix algebra

$$M_n(\mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in \mathbb{C} \right\}$$

If $H =$ Hilbert space

$$B(H) = \{ T \colon H \to H \mid T \text{ is linear and continuous } \}$$

$B(H)$ is an *algebra*: we can add $T + S$ and compose operators TS, and multiply by scalars λT, so that all the fundamental properties of arithmetics hold ...

but the commutativity law!! in general

$$TS \neq ST$$

We have an involution $T \mapsto T^*$ in $B(H)$, defined by means of the *adjoint operator*:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

$x, y \in H$.

In addition, we have a norm $\|\cdot\|$ in $B(H)$:

$$\|T\| = \sup_{\|x\|\leq 1}\|Tx\|.$$

and a good partnership $* \leftrightarrow$ norm :

$$\|T^*T\| = \|T\|^2$$

There are various topologies of interest in $B(H)$. We will only mention two of them:

The *uniform topology*:

$$T_n \to T \iff \|T - T_n\| \to 0$$

The *strong operator topology*:

$$T_n \to_s T \iff T_n(h) \to T(h) \quad \forall h \in H.$$

Remark

If $\{E_n\}$ is a sequence of orthogonal projections with $E_n E_m = 0$ for $n \neq m$, then $\sum_{n=1}^{\infty} E_n$ converges in the strong operator topology, but NOT in the uniform topology.

Definition

A **von Neumann algebra** is a $*$-subalgebra $\mathcal{M}$ of $B(H)$ which is closed in the strong operator topology.

Definition

A C^*-**algebra** is a $*$-subalgebra of $B(H)$ closed in the uniform topology.

Every von Neumann algebra is a C^*-algebra, but the converse is not true.

For $X \subseteq B(H)$, define the *commutant* X' of X as

$$X' = \{y \in B(H) \mid yx = xy \ \forall x \in X\}.$$

Theorem (von Neumann's Bicommutant Theorem, 1929)

A $*$-subalgebra $\mathcal{M}$ of $B(H)$ is a von Neumann algebra if and only if

$$\mathcal{M} = \mathcal{M}''.$$

This says that for each von Neumann algebra $\mathcal{M}$ we have another von Neumann algebra $\mathcal{M}'$, since $\mathcal{M}' = \mathcal{M}'''$.

Discrete and continuous factors

Murray and von Neumann performed a deep study of *factors*: von Neumann algebras $\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{M}' = \mathbb{C} \cdot 1$.

$M_n(\mathbb{C})$ and $B(H)$ (H ∞-dim.) are *discrete factors*:
the dimensions of the projections take **all the values in** $\{0, 1, \ldots, n\}$ and $\mathbb{N} \cup \{\infty\}$ respectively.

M-vN : *continuous factors*: $\mathcal{M}$ such that

$$\dim(\mathrm{Proj}(\mathcal{M})) = [0, 1]$$

filling all the interval.

Example

Let $\mathbb{F}_2$ be the free group on two generators a, b, $H = \ell^2(\mathbb{F}_2)$ the Hilbert space with orthonormal basis $\mathbb{F}_2$.

$$\mathcal{N}(\mathbb{F}_2) = \{ T \in B(H) \mid T(\xi x) = T(\xi)x \ \ \forall \xi \in H, \forall x \in \mathbb{F}_2 \}$$

is a continuous factor.

The dimension of a projection $E \in \mathcal{N}(\mathbb{F}_2)$ is computed as follows:

$$\dim(E) = \langle E(e), e \rangle \in [0, 1]$$

where e is the neutral element of $\mathbb{F}_2$.

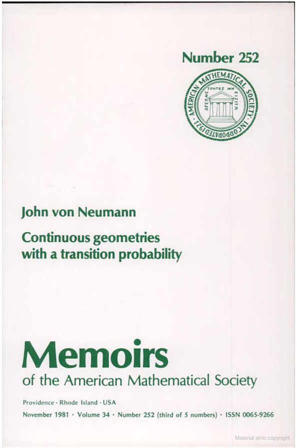
Type Classification

Murray and von Neumann established the following classification of factors:

- type I_n: $M_n(\mathbb{C}) \longleftrightarrow \dim(\text{Proj}(M_n(\mathbb{C}))) = \{0, \ldots, n\}$
- type I_∞: $B(H) \longleftrightarrow \dim(\text{Proj}(B(H))) = \mathbb{N} \cup \infty$
- type II_1: $\longleftrightarrow \dim(\text{Proj}(\mathcal{M})) = [0, 1]$
- type II_∞: $\longleftrightarrow \dim(\text{Proj}(\mathcal{M})) = [0, \infty]$
- type III: $\longleftrightarrow \dim(\text{Proj}(\mathcal{M})) = \{0, \infty\}$

Related works by von Neumann

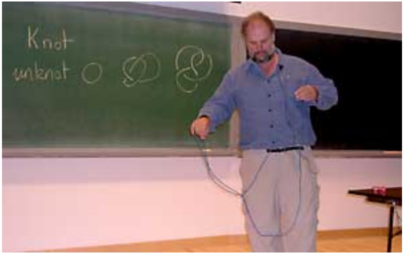
- Lattice theory. Continuous geometries.
- Von Neumann regular rings ≡ rings which coordinatize the continuous geometries.



- Knot theory (Vaughan Jones' polynomial)
- Non-commutative geometry (Alain Connes)

Knot theory (The Jones polynomial)

Vaughan Jones initiated the study of subfactors and applied techniques from operator algebras to the theory of knots. He was awarded the Fields Medal in 1990.



The Jones polynomial and operator algebras

The *Jones polynomial* $V_L(t)$ of a knot L is a Laurent polynomial $V_L(t) \in \mathbb{Z}[t, t^{-1}]$ which is quite powerful at distinguishing knots one from another.

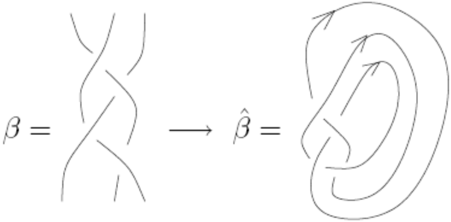
V  $= 1$

V  $= t + t^3 - t^4$

V  $= \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$

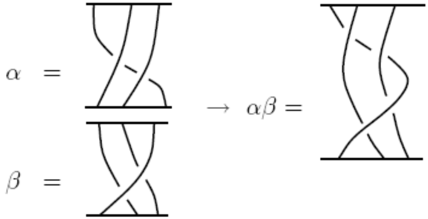
Jones defined $V_L(t)$ using operator algebras.

Every knot comes from a *braid*:



The braid group B_n is the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \;\; \sigma_i \sigma_j = \sigma_j \sigma_i \text{ si } |i-j| \geq 2$$



Jones defined a representation ρ_n of the braid group B_n on a von Neumann algebra A_n, endowed with a particular *trace function* $\mathrm{tr}\colon A_n \to \mathbb{C}$.

Note

The characteristic property of a trace is $\mathrm{tr}(ab) = \mathrm{tr}(ba)$.

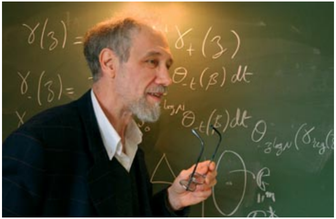
Jones defined $V_{\hat{\alpha}}(t)$ by the formula:

$$V_{\hat{\alpha}}(t) = \Big(- \sqrt{t} - \frac{1}{\sqrt{t}} \Big)^{n-1} \sqrt{t}^{-e} \mathrm{tr}(\rho_n(\alpha))$$

where e is the exponent sum of $\alpha \in B_n$ as a word in $\sigma_1, \ldots, \sigma_{n-1}$.

Non-commutative geometry

Alain Connes has applied the tools from operator algebras to the study of many problems related with differential geometry, physics, number theory ...



Some ideas:

Theorem (Gelfand-Naimark)

The commutative C^*-algebras (with unit) are the ones of the form $C(X)$ with X a compact Hausdorff space.

C^*-algebra theory $\leftrightarrow$ non-commutative topology

Theorem

The commutative von Neumann algebras are exactly the algebras $L^\infty(X, \mu)$, where μ is a measure on X.

von Neumann algebra theory $\leftrightarrow$ non-commutative measure theory

Connes refined the type classification obtained by Murray and von Neumann. He was awarded the Fields Medal in 1982.

In particular Connes discovered a classification for type III factors, in factors of types III_λ, where $\lambda \in [0,1]$.

This number λ is defined through certain one-parameter groups of automorphisms

$$\sigma^{\varphi} \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{M}),$$

$$\sigma_t^{\varphi}(a) = \Delta_{\varphi}^{-it} a \Delta_{\varphi}^{it}$$

associated to positive faithful functionals φ defined on $\mathcal{M}$, where Δ_φ is a certain modular operator associated to $(\mathcal{M}, \varphi)$.

Note

Indeed, Connes showed that σ^{φ} does not depend on φ *modulo inner automorphisms*.

Theorem (Connes, 1973)

Let $\lambda \in (0,1)$.
(i) Let $\mathcal{M}$ be a type III_λ factor. Then there is a factor $\mathcal{N}$ of type II_∞ and $\theta \in \mathrm{Aut}(\mathcal{N})$ such that $\tau \circ \theta = \lambda\tau \;\forall$ trace τ on $\mathcal{N}$. Moreover

$$\mathcal{M} \cong \mathcal{N} \rtimes_\theta \mathbb{Z}.$$

(ii) Let $\mathcal{N}$ be a type II_∞m factor and $\theta \in \mathrm{Aut}(\mathcal{N})$ such that $\tau \circ \theta = \lambda\tau \;\forall$ trace τ on $\mathcal{N}$. Then $\mathcal{N} \rtimes_\theta \mathbb{Z}$ is a type III_λ factor.

Definition

A unital C^*-algebra A is said to be **purely infinite simple** if $A \neq \mathbb{C}$ and for any $a \neq 0$ in A there are $x, y \in A$ such that

$$xay = 1.$$

A purely infinite simple $\leftrightarrow$ A has a lot of projections, and all them are "infinite" and "comparable" to 1.

Definition

A C^*-algebra is said to be **separable** if it contains a countable dense subset (in the uniform topology)

Theorem (Kirchberg-Phillips, 2000)

The purely infinite simple separable C^*-algebras are classified *up to isomorphism* by their K-theory. If A and B are purely infinite simple separable and

$$K_0(A) \cong K_0(B), \quad K_1(A) \cong K_1(B)$$

then there is an isomorphism $\varphi \colon A \to B$ which induces the above isomorphisms in K-theory.

The type III factors are always purely infinite simple but they are NEVER separable. In fact the above theorem does not apply here because always $K_0(\mathcal{M}) = 0 = K_1(\mathcal{M})$ if $\mathcal{M}$ is a type III factor.

George Elliott has established a Classification Programme for separable C^*-algebras by K-theoretical invariants. This programme has attracted (an is still attracting) the attention of many researchers worldwide.

Thank you very much!!